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A characterization of infinite planar primitive graphs

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Abstract

It is shown that the automorphism group of an infinite, locally finite, planar graph acts primitively on its vertex set if and only if the graph has connectivity 1 and, for some integer $m \geqslant 2$, every vertex is incident with exactly m lobes, all of which are finite. Specifically, either all of the lobes are isomorphic to K_4 or all are circuits of length p for some odd prime p.

MSC: 05C25; 05C75; 20B15

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1. Introduction

A graph is said to be *primitive* if its automorphism group acts as a primitive permutation group on its vertex set. A *lobe* of a graph Γ is a subgraph of Γ induced by an equivalence class of edges of Γ with respect to the equivalence relation of lying on a common circuit. The purpose of this paper is to prove the following result.

Theorem 1.1. *An infinite, locally finite, planar graph Γ is primitive if and only if it has connectivity exactly* 1 *and there exists an integer $m \geqslant 2$ such that every vertex of Γ is incident with exactly m lobes. Moreover, either all of the lobes are isomorphic to K_4 or all are circuits of length p for some fixed odd prime p.*

In Section 2, we supply the needed vocabulary and list the results from other sources as well as some elementary facts that will be invoked in the course of the

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proof of Theorem 1.1. Since the sufficiency of the condition is straightforwardly handled, most of this article is concerned with the proof of its necessity. In Section 3, we state four additional results needed to prove Theorem 1.1 (deferring their rather technical proofs to Sections 4 and 5) and then proceed to give the proof of our main result.

2. Background results and terminology

Let V be a nonempty set and let G be a group of permutations on V. If $\alpha \in G$ and $W \subseteq V$, then $\alpha[W] = \{\alpha(x) : x \in W\}$. We say that G *acts transitively* on V if for all $x, y \in V$, there exists $\alpha \in G$ such that $\alpha(x) = y$. For $x \in V$, the *stabilizer* of x is the subgroup $G_x = \{\alpha \in G : \alpha(x) = x\}$. A subset $B \subseteq V$ is called a *block* (of imprimitivity with respect to G) if for all $\alpha \in G$, either $\alpha[B] = B$ or $\alpha[B] \cap B = \emptyset$. Clearly $\emptyset$, V, and the singleton subsets of V are blocks. If G acts transitively and admits only these so-called *trivial* blocks, then G is *primitive*; if G is transitive but admits nontrivial blocks, then G is *imprimitive* on V.

Proposition 2.1 (Jung and Watkins [15, Lemma 4.1]). *Let G be a group of permutations on a set V. Let $B_0 \subset V$ and for each nonnegative integer n, define*

$$B_{n+1} = \bigcup \{\alpha[B_0] : \alpha \in G;\ \ \alpha[B_0] \cap B_n \neq \emptyset\}.$$

Then $\bigcup_{n=0}^{\infty} B_n$ is the smallest block containing B_0.

Throughout this article, graphs and their subgraphs will be denoted by capital Greek letters. We let $V(\Gamma)$, $E(\Gamma)$, and $\kappa(\Gamma)$ denote, respectively, the vertex set, the edge set, and the connectivity of Γ. If $U \subseteq V(\Gamma)$, then $\langle U \rangle$ denotes the subgraph of Γ induced by the set U. The *valence* of a vertex x is the number of edges incident with it and will be denoted by $\rho(x)$. If $\rho(x) = r$ for all $x \in V(\Gamma)$, we say that Γ is *r-valent* and write $\rho(\Gamma) = r$. The graphs that we consider may be finite or infinite, but without exception—*and usually without further mention*—they are *locally finite*, i.e., all vertices have finite valence.

A graph Γ is said to be *vertex-transitive* or *primitive* or *imprimitive* if its automorphism group $\mathrm{Aut}(\Gamma)$ acts, respectively, transitively or primitively or imprimitively on $V(\Gamma)$. Γ is *edge-transitive* if $\mathrm{Aut}(\Gamma)$ acts transitively on $E(\Gamma)$. A *block* (of Γ) is a subset of $V(\Gamma)$ that is a block of imprimitivity with respect to $\mathrm{Aut}(\Gamma)$ acting on $V(\Gamma)$. The finite planar primitive graphs were characterized in [5] as follows.

Proposition 2.2. *A finite planar graph is primitive if and only if it is one of the following*: (i) *an edgeless graph*, (ii) K_2, (iii) *a circuit of odd prime length*, (iv) K_4.

A *lobe* of a graph Γ is a subgraph of Γ induced by an equivalence class of edges of Γ with respect to the equivalence relation of lying on a common circuit. Thus any two distinct lobes have at most one vertex in common. (We prefer to use the term "lobe" (see [18, p. 86]) to the frequently used term "block" in order to avoid ambiguity when we refer to a block of imprimitivity of Γ.)

Primitive (though not necessarily planar) graphs of connectivity 1 admit the following characterization.

Proposition 2.3 (Jung and Watkins [15, Theorem 4.2]). *Let Γ be an infinite graph with $\kappa(\Gamma) = 1$. Γ is primitive if and only if all of the following hold*:

(a) *there exists an integer $m \geqslant 2$ such that every vertex is incident with exactly m lobes*;
(b) *the lobes of Γ are pairwise isomorphic*;
(c) *the lobes are themselves primitive graphs not isomorphic to K_2.*

Our main result, Theorem 1.1, states that the infinite planar primitive graphs are exactly those given in Proposition 2.3 but where the lobes must all be one of the two 2-connected candidates given in Proposition 2.2. The sufficiency of the condition in Theorem 1.1 is immediate from these two propositions. Thus our mission for the remainder of this article lies clearly before us: to eliminate all other possibilities. The following observation gets us off to an easy start.

Proposition 2.4. *Let Γ be a vertex-transitive graph.*

(a) *If Γ is not connected and has nontrivial components, then it is imprimitive.*
(b) *If Γ is bipartite, then it is imprimitive.*

Denote by $d(-,-)$ the usual metric on a connected graph Γ. For each nonnegative integer n let $f(n, x_0)$ denote the number of vertices at distance at most n from some arbitrary vertex x_0. If Γ is vertex-transitive, then clearly $f(n, x_0)$ is independent of x_0, and so we write simply $f(n)$. We say that a connected infinite graph Γ has *exponential growth* if there exists a constant $a > 1$ such that $\liminf_{n\to\infty}(f(n)/a^n) > 0$. The graph Γ has *polynomial growth* and, more specifically, its *growth degree* is k if for some constant $c > 0$ we have $k := \inf\{k' : f(n) \leqslant cn^{k'} \text{ for all } n > 0\}$. That k is always an integer follows from a result by Gromov (see [7]). A graph with *linear* growth has growth degree 1; if the growth degree is 2, then its growth is *quadratic*. Next, we eliminate graphs that "grow" too slowly.

Proposition 2.5 (Godsil et al. [4, Theorem 4]). *Let Γ be an infinite, (locally finite,) connected, vertex-transitive graph. If Γ has polynomial growth, then Γ is imprimitive.*

We use the notion of an "end" as formulated by Halin [10]. For a locally finite graph Γ, there is a simpler definition: the number $\varepsilon(\Gamma)$ of ends of Γ is the supremum of the number of infinite components of $\Gamma - \Phi$ as Φ ranges over all finite subgraphs of Γ. If Γ is vertex-transitive, then it is well known that $\varepsilon(\Gamma) = 1$, 2 or $2^{\aleph_0}$ (see [11,14]).

Proposition 2.6 (Imrich and Seifter [13, Theorem 2.8 and Proposition 3.1]). *Let Γ be a vertex-transitive connected graph. Then Γ has linear growth if and only if it has exactly two ends.*

Combining this result with Proposition 2.5, we may now eliminate vertex-transitive graphs Γ for which $\varepsilon(\Gamma) = 2$. We see next that the same holds for $\kappa(\Gamma)$.

Proposition 2.7 (Jung and Watkins [16]). *Let n be a nonnegative integer. There exists an infinite primitive graph Γ with $\kappa(\Gamma) = n$ if and only if $n \neq 2$.*

We may now focus our attention on infinite planar graphs that are 3-connected and hence have an essentially unique planar embedding (see [12]). Thus we may identify such a graph Γ with the underlying graph of a planar map and speak of the set $F(\Gamma)$ of *faces* of Γ. If $f \in F(\Gamma)$, then the *covalence* $\rho^*(f)$ of f is the number of edges (or vertices) incident with f. Two faces (or two vertices) are *adjacent* if they are incident with a common edge.

3. Proof of the main result

In order to prove Theorem 1.1, we require four more results. Taken together the two theorems will imply the imprimitivity of edge-transitive, 3-connected, planar maps; their proofs are rather technical and are deferred to Section 4. The two lemmas pertain to the kinds of lobes that a planar primitive graph of connectivity 1 may have; their proofs are deferred to Section 5.

Theorem 3.1. *All* 1-*ended*, *vertex-transitive*, *edge-transitive*, *planar graphs are imprimitive*.

Theorem 3.2. *All infinitely-ended*, *vertex-transitive*, *edge-transitive*, 3-*connected*, *planar graphs are imprimitive. In particular*, *the group generated by all of the vertex-stabilizers is the stabilizer of a nontrivial block of imprimitivity*.

Suppose that the edge-orbits induced by the action of $\mathrm{Aut}(\Gamma)$ on the graph Γ in turn induce the subgraphs $\Theta_1, \ldots, \Theta_k$. These edge-transitive subgraphs are called the *layers* of Γ.

Proposition 3.3. *Let Θ be a layer of a primitive graph Γ. Then Θ is primitive and hence a connected spanning subgraph of Γ.*

Proof. This follows immediately from Proposition 2.4(a) and the fact that $\mathrm{Aut}(\Gamma) \leqslant \mathrm{Aut}(\Theta)$. □

Lemma 3.4. *Let Γ be an infinite, primitive graph that is not edge-transitive. If one of its layers has connectivity exactly* 1 *with lobes that are isomorphs of K_4, then Γ is not planar.*

Lemma 3.5. *Let Γ be a vertex-transitive graph that is not edge-transitive. Suppose that two of the layers of Γ are primitive subgraphs whose lobes are circuits of odd prime length. Then Γ is not planar.*

We now present the proof of our main result.

Proof of Theorem 1.1. The sufficiency of the condition for primitivity follows immediately from Propositions 2.3 and 2.2.

To prove the necessity of the condition, we assume that Γ is an infinite, planar, primitive graph.

Suppose that Γ is not edge-transitive, and let Θ be a layer of Γ. Suppose that $\kappa(\Theta) > 1$. Then Θ is primitive by Proposition 3.3 and hence is 3-connected by Proposition 2.7. However, by Propositions 2.6 and 2.5, Θ would be imprimitive in the case that $\varepsilon(\Theta) = 2$ and would be imprimitive otherwise by Theorems 3.1 and 3.2. Since Θ is connected by Proposition 3.3, we have $\kappa(\Theta) = 1$, and we consider a lobe Λ of Θ. If some automorphism of Θ maps an edge of Λ onto an edge of Λ, then it must fix Λ. Since Θ is edge-transitive, this implies that so is Λ.

By Proposition 2.3, Λ must be primitive, too, and hence 3-connected if it is infinite. Application to Λ of the same results cited in the previous paragraph leads to a contradiction. It follows that all lobes of all layers are finite.

By Propositions 2.2 and 2.3, the lobes of Θ either are all isomorphic to K_4 or are all odd circuits of the same odd prime length. If the lobes of Θ are isomorphic to K_4, then $\Gamma = \Theta$ by Lemma 3.4. Lemma 3.5 yields the same conclusion if the lobes of Θ are circuits of odd prime length. The rest now follows by Proposition 2.3. □

4. Edge-transitive, 3-connected, planar graphs

The goal of this section is to prove Theorems 3.1 and 3.2. These results, together with Propositions 2.6 and 2.5, will imply that all infinite vertex- and edge-transitive, 3-connected, planar graphs are imprimitive. To each such map we associate a triple $\langle r; c_1, c_2 \rangle$ called its *symbol*, where r denotes its (constant) valence, and c_1 and c_2 denote its (not necessarily distinct) covalences. Edge-transitivity implies that around each vertex, the covalences of its incident faces alternate between the two values c_1 and c_2. When $c_1 = c_2$, we let c denote the constant covalence. Clearly if r is odd, then $c_1 = c_2$. The following results can be found in [8,9].

Proposition 4.1. *Let Γ be the underlying graph of an r-valent planar map such that about each vertex the covalences of the incident faces are alternately c_1 and c_2 (possibly $c_1 = c_2$). Then*

(a) *Γ is vertex- and edge-transitive.*
(b) *If Γ is 1-ended, then it is unique (up to isomorphism), i.e., Γ is uniquely determined by the triple $\langle r; c_1, c_2 \rangle$.*

(c) *Let*

$$Q = \frac{2}{r} + \frac{1}{c_1} + \frac{1}{c_2}.$$

If $Q > 1$, *then* Γ *is finite*; *if* $Q = 1$, *then* Γ *has linear or quadratic growth*; *if* $Q < 1$, *then* Γ *has exponential growth*.

We also have use for the following result.

Proposition 4.2 (Bonnington et al. [2, Lemma 2.2]). *If a graph is vertex-transitive and* 1-*ended*, *then it is* 3-*connected*.

Let Γ be a vertex- and edge-transitive, 3-connected, planar map. Let $x \in V(\Gamma)$ and let $f_0, \ldots, f_{r-1}$ (subscripts to be read in $\mathbb{Z}_r$) denote the faces incident with x in cyclic order around x. If r is even, then *the face opposite* f_i *across* x is the face $f_{i+r/2}$. If r is odd, then *the edge opposite* f_i *across* x is the edge incident with the two faces $f_{i+(r-1)/2}$ and $f_{i+(r+1)/2}$. Dually, let $x_0, \ldots, x_{c-1}$ denote the vertices around a c-covalent face f, and let g_i denote the face adjacent to f and incident with the edge $[x_{i-1}, x_i]$ (subscripts to be read in $\mathbb{Z}_c$). Since Γ is 3-connected, these faces are all distinct. If c is even, we may define the *vertex opposite* x_i *across* f and the *face opposite* g_i, *across* f to be $x_{i+c/2}$ and $g_{i+c/2}$, respectively. If c is odd, then the vertex x_i and the face $g_{i+(c+1)/2}$ are *opposite across* f.

We designate by $\mathcal{G}_{a,b}$ the class of all 1-ended, 3-connected, planar graphs X such that $a \leqslant \rho(v) < \infty$ for all $v \in V(X)$ and $b \leqslant \rho^*(f) < \infty$ for all $f \in F(X)$.

The following manner of labeling locally finite planar maps is originally due to Bilinski [1] and has been used frequently in [9]. Let Γ be an infinite, locally finite, 1-ended, planar, 3-connected graph, and let $x \in V(\Gamma)$. Γ may be regarded as the underlying graph of a *Bilinski diagram M* in the light of the following notation:

$U_0 = \{x\}$; x will be called the *center* of Γ.

F_1 is the set of faces incident with x.

For $r \geqslant 1$, U_r is the set of those vertices not in U_{r-1} that are incident with a face in F_r.

For $r \geqslant 1, F_{r+1}$ is the set of faces not in F_r that are incident with a vertex in U_r.

Using the above terminology, a vertex in $U_i (i \geqslant 1)$ is called an ℓ-*vertex* if it is adjacent to exactly ℓ vertices in U_{i-1}.

Proposition 4.3. *Let the* 1-*ended*, 3-*connected*, *planar map* Γ *be presented as a Bilinski diagram. Let* $i \geqslant 1$.

(a) *If* $\Gamma \in \mathcal{G}_{6,3} \cup \mathcal{G}_{4,4} \cup \mathcal{G}_{3,6}$, *then the subgraph* $\langle U_i \rangle$ *is a circuit.*
(b) *If* $\Gamma \in \mathcal{G}_{4,4} \cup \mathcal{G}_{3,6}$, *then* $\ell \leqslant 1$ *for all* ℓ *such that* U_i *contains an* ℓ-*vertex.*
(c) *If* $\Gamma \in \mathcal{G}_{6,3}$, *then* $\ell \leqslant 2$ *for all* ℓ *such that* U_i *contains an* ℓ-*vertex.*
(d) *If* $\Gamma \in \mathcal{G}_{6,3} \cup \mathcal{G}_{4,4}$ *and* $f \in F_i$, *then* f *is incident with at most one edge of* $\langle U_{i-1} \rangle$.
(e) *If* $\Gamma \in \mathcal{G}_{3,6}$ *and* $f \in F_i$, *then* f *is incident with at most two edges of* $\langle U_{i-1} \rangle$.

The statements in this proposition involving $\mathcal{G}_{4,4}$ are to be found in [17]; the remaining parts are due to Bruce [3].

Proof of Theorem 3.1. Suppose that Γ is vertex-transitive, edge-transitive, planar, and 1-ended, with symbol $\langle r; c_1, c_2 \rangle$. By Proposition 4.2, Γ is 3-connected.

Case 1: *Both* c_1 *and* c_2 *are even*. Since Γ is 1-ended, its cycle space is generated by the circuits that are the boundaries of its faces (cf. [19, Theorem 7.4]). This implies that Γ is bipartite and therefore imprimitive by Proposition 2.4(b).

In all of the remaining cases we will regard Γ as a Bilinski diagram with center x_0. The face f_1 will belong to F_1. In each case, a second vertex y_0 will be defined accordingly. Then, in the notation of Proposition 2.1, where $\mathrm{Aut}(\Gamma)$ acts on $V(\Gamma)$, we will let $B_0 = \{x_0, y_0\}$ and define B_n inductively, as well as the limit set B, exactly as in the Proposition. Thus B will be a block (of imprimitivity) of Γ, and we will need to argue in each case that B is a proper subset of $V(\Gamma)$, i.e., a nontrivial block. The reader is referred to Fig. 1 for some of these cases and their subcases. In Fig. 1, vertices in B are indicated by large black disks.

Case 2: *Exactly one of* c_1 *and* c_2 *is even*. Thus $r \geqslant 4$. Without loss of generality, assume that c_1 is even. Let f_1 be a c_1-covalent face in F_1, and let y_0 be the vertex opposite x_0 across f_1. Note that every automorphism of Γ must map c_1-covalent faces onto c_1-covalent faces and pairs of opposite vertices across c_1-covalent faces onto other such pairs. Since $c_1 \geqslant 4$, clearly y_0 is a 0-vertex, and any other c_1-covalent face f_2 incident with y_0 belongs to F_2 and is incident with no vertex in U_1 other than y_0. Thus the vertex opposite y_0 across f_2 is a 0-vertex in $B_1 \cap U_2$. It is easy to see inductively that $B_n \backslash B_{n-1}$ is a proper subset of U_{n+1} for all $n \geqslant 1$, and so the block of imprimitivity B is a proper subset of $V(\Gamma)$.

In all the remaining cases, both c_1 and c_2 are assumed to be odd and, without loss of generality, $c_1 \geqslant c_2$. These cases will exhaust all remaining combinations of values for r, c_1, and c_2 that, according to Proposition 4.1, imply exponential growth.

Case 3: *One of the following holds*:

Subcase 3a: $r = 3$ (and hence $c_1 = c_2 \geqslant 7$).

Subcase 3b: $r \geqslant 4$ and $c_1, c_2 \geqslant 5$.

Let $\{h, k\} = \{1, 2\}$, let f_1 be c_h-covalent, and let f_2 be the face opposite x_0 across f_1. Thus $f_2 \in F_2$ is c_k-covalent. Let y_0 be the vertex opposite f_1 across f_2. Thus $y_0 \in U_2$; y_0 is a 0-vertex. Recall that always $B_0 = \{x_0, y_0\}$. Note that under the action of $\mathrm{Aut}(\Gamma)$, x_0 can be mapped onto y_0 so that f_1 is mapped onto any given c_h-covalent face incident with y_0. Similarly y_0 can be mapped onto x_0 so that f_2 is mapped onto any given c_k-covalent face incident with x_0 (cf. [6, Corollary 3.6]). The "minimal" instance of subcase 3b is sketched in Fig. 1, from which it is obvious that the limiting set B is a proper subset of $V(\Gamma)$. The details are left to the reader.

Case 4: $c_1 = c_2 = 3$. Let f_1 and f_2 be as in Case 3, but let z_1 be the vertex opposite f_1 across f_2.

First suppose that r is odd, and so $r \geqslant 7$ and $\Gamma \in \mathscr{G}_{6,3}$. Let z_2 be the vertex such that $[z_1, z_2]$ is the edge opposite f_2 across z_1. Let f_3 be the face opposite $[z_1, z_2]$ across z_2, let f_4 be the face opposite z_2 across f_3, and let y_0 be the vertex opposite f_3 across f_4. Thus $f_i \in F_i (i = 1, \ldots, 4)$ and $y_0 \in U_5$.

Suppose that for some $n \geqslant 1$, $i \geqslant 2$, and $\alpha \in \mathrm{Aut}(\Gamma)$, we have $\alpha(x_0) \in U_i$ and $\alpha[B_0] \cap B_n = \{\alpha(x_0)\}$. (By the definition in Proposition 2.1, $\alpha(y_0) \in B_{n+1}$.) Suppose

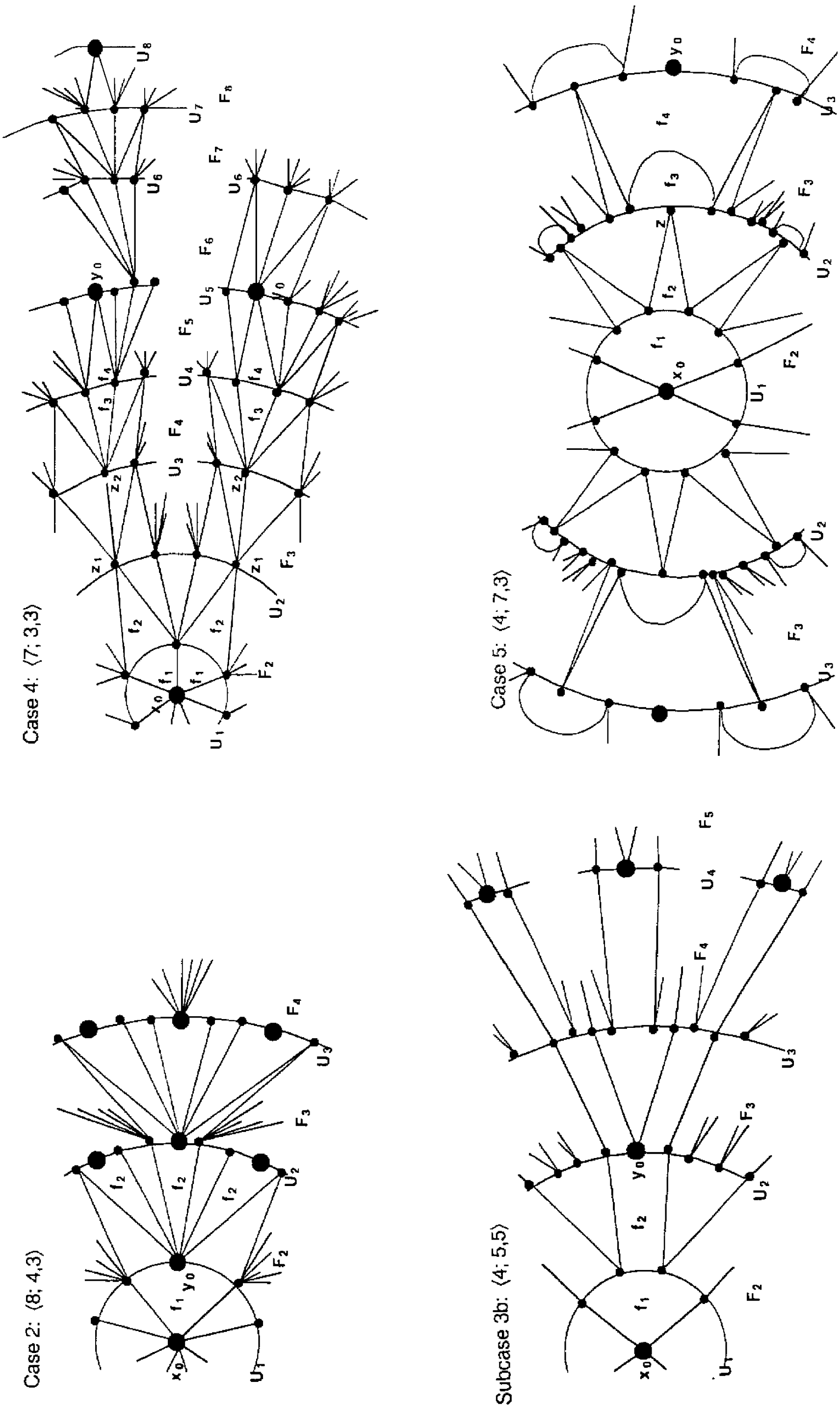

Fig. 1. Proof of Theorem 3.1.

also that $\alpha(y_0) \in U_j$ for some $j < i$ and that we have chosen n to be the least value such that this is so. Then $\alpha(x_0) = \beta(y_0)$ for some $\beta \in \mathrm{Aut}(\Gamma)$ such that $\beta[B_0] \cap B_{n-1} = \{\beta(x_0)\}$.

Exactly two of the faces in F_i incident with $\alpha(x_0)$ are also incident with an edge of $\langle U_i \rangle$. Suppose that $\beta(f_4)$ is such a face. Then $\beta(f_3)$ is also in F_i, and there are two possibilities to consider: $\beta(f_3)$ is incident with an edge in $\langle U_i \rangle$ or with an edge in $\langle U_{i-1} \rangle$ (and clearly not both).

If $\beta(f_3)$ is incident with an edge in $\langle U_i \rangle$, then we have $\beta(z_2) \in U_i$. By Proposition 4.3(c), $\beta(z_2)$ is at most a 2-vertex. Since $r \geqslant 7$, this implies that $\beta(z_1) \in U_{i+1}$. Applying this argument to $\beta(z_1)$ forces $\beta(f_2) \in F_{i+2}$ and eventually $\beta(x_0) \in U_{i+2}$, contrary to the minimality of n. If $\beta(f_3)$ is incident with an edge in $\langle U_{i-1} \rangle$, then we have $\beta(z_2) \in U_{i-1}$ and $\beta(z_1) \in U_{i-1} \cup U_i$ because $\beta(z_2)$ is at most a 2-vertex. This forces $\beta(f_2)$ to be in $F_i \cup F_{i+1}$ and eventually $\beta(x_0) \in U_i \cup U_{i+1} \cup U_{i+2}$.

We have shown that for all $\beta \in \mathrm{Aut}(\Gamma)$, the face $\beta(f_4)$ must be that unique face in F_{i-1} induced by $\alpha(x_0)$ and the two vertices in U_{i-1} adjacent to $\alpha(x_0)$. Hence $\alpha(f_1) \in F_{i+1}$ and $\alpha(z_1) \in U_{i+1} \cup U_{i+2}$, whence clearly $j > i$, which is in fact stronger than needed.

When r is even, the argument is even simpler; moreover, $r \geqslant 8$. Now we don't need z_2, as there exists a face f_3 opposite f_2 across z_1. Then choose f_4 and y_0 exactly as when r is odd. The proof continues almost identically, and we conclude that $j \geqslant i$.

Case 5: $\langle 4; c_1, 3 \rangle$. Thus $c_1 \geqslant 7$. Let f_1 be c_1-covalent and let f_2 be the face opposite x_0 across f_1. Thus $f_2 \in F_2$ and $\rho^*(f_2) = 3$. Let z be the vertex opposite f_1 across f_2 and let f_3 be the face opposite f_2 across z. Thus $f_3 \in F_3$ and $\rho^*(f_3) = 3$. (Note that Proposition 4.3(a) does not apply here, as $\Gamma \in \mathscr{G}_{4,3}$. In fact, all three edges incident with f_3 belong to $\langle U_2 \rangle$.) Finally, let f_4 be the (unique) c_1-covalent face in F_3 adjacent to f_3 and let y_0 be the vertex opposite f_3 across f_4. Thus $y_0 \in U_3$.

Clearly there exist exactly two automorphisms of Γ that interchange x_0 and y_0 (one being a rotation and the other a reflection), and both map $f_j \leftrightarrow f_{3-j} (j = 1, \ldots, 4)$. If $\alpha \in \mathrm{Aut}(\Gamma)$ and $\alpha(x_0) = y_0$ but $\alpha(y_0) \neq x_0$, then $\alpha(y_0) \in U_6$. One easily verifies that for each $n \geqslant 1$, B_n contains exactly two vertices from U_{3i} for $1 \leqslant i \leqslant n$.

Case 6: $\langle r; c_1, 3 \rangle$, *where* $r \geqslant 6$ *and* $c_1 \geqslant 5$. Again $\Gamma \in \mathscr{G}_{6,3}$. Let f_1 be c_1-covalent and let f_2 be the face opposite x_0 across f_1. Thus $\rho^*(f_2) = 3$ and we let f_3 be either of two faces in F_2 adjacent to f_2. Finally, y_0 is the vertex opposite f_2 across f_3. Note that y_0 is a 0-vertex in U_2. If $\alpha \in \mathrm{Aut}(\Gamma)$ and $\alpha(x_0) \neq y_0$, then $\alpha(y_0) \in U_4$ and is also a 0-vertex (see Fig. 2 for the "minimal" instance $\langle 6; 5, 3 \rangle$). One easily verifies in a manner similar to that of Case 5 that for each $n \geqslant 1$, B_n contains exactly $4^{i-1} r$ vertices from U_{2i} for $1 \leqslant i \leqslant n$ and no vertices at all from U_i when i is odd. □

Altogether different techniques are required to show that the analogue of Theorem 3.1 also holds for infinitely-ended maps. Bilinski diagrams of infinitely-ended maps lack the orderliness of the 1-ended maps. Except for small values of i, the subgraphs $\langle U_i \rangle$ are not circuits and are not even connected. In fact, $\langle U_i \rangle$ may have arbitrarily many components if i is sufficiently large. A further difficulty with infinitely-ended maps comes from the fact that their automorphism groups are

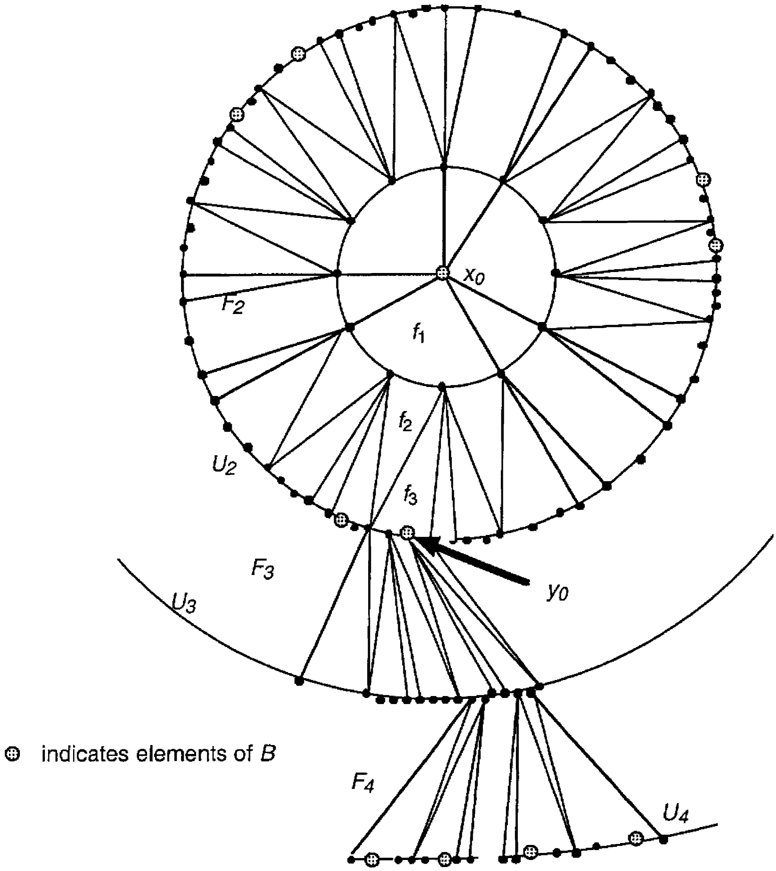

Fig. 2. Proof of Theorem 3.1, Case 6.

"smaller" in the sense that the range of choices of automorphisms that send a given vertex onto another given vertex (or edge-to-edge or face-to-face) is strictly smaller than for a 1-ended map with the same parameters $\langle r; c_1, c_2 \rangle$. For this latter reason, we present for completeness a brief discussion of the automorphism group structure of edge-transitive, infinitely-ended, 3-connected, planar maps.

A *corner* of a map Γ is an ordered pair $(x,f) \in V(\Gamma) \times F(\Gamma)$ where x and f are incident. Clearly there are exactly two edges that are incident with each corner. A *flag* is a triple $(x,e,f) \in V(\Gamma) \times E(\Gamma) \times F(\Gamma)$ where e is incident with the corner (x,f). If Γ is a 3-connected planar map and $\alpha \in \mathrm{Aut}(\Gamma)$, then α is uniquely determined by its action on any flag of Γ (cf. [6, Lemma 3.1]).

The map-automorphism (which is unique if it exists) that fixes (x,f) but interchanges the two edges incident with (x,f) is called a *corner reflection* and will be denoted by θ_{xf}. Given a face f, corner reflections extend to map-automorphisms at corners incident with f according to only three possibilities: at all of the corners, exactly at alternate corners, or at none of the corners. Clearly if $\rho^*(f)$ is odd, then the option of alternate corners cannot apply. Dually, an analogous situation holds for the $\rho(v)$ corners incident with any given vertex v.

The automorphism of order $r/\mathrm{lcd}(r,j)$ that rotates Γ counterclockwise about a vertex x by j edges (and j faces) will be denoted by σ_x^j. Dually, the rotation about a face f will be denoted by σ_f^j.

A *Petrie walk* Π has the property that every two consecutive edges of Π are incident with a common face but no three consecutive edges have this property. Every Petrie walk in an edge-transitive, 3-connected, planar map is either a circuit of even length (called a *Petrie circuit*) or a double ray ([6, Theorem 5.3]). The automorphism (again, unique if it exists) that advances Π along itself in some given direction by one edge is called a *glide reflection* and is denoted by γ_Π.

The action upon a map by a vertex- or face-rotation preserves its orientation. The action of a corner reflection or a glide reflection reverses orientation. A map-automorphism reverses orientation if and only if it can be expressed as the composition of an odd number of reflections (and any number whatever of rotations).

Stabilizers of vertices, faces, and Petrie walks are always subgroups of a dihedral group D_n of order $2n$, where n equals accordingly the valence of a vertex, the covalence of a face, or the length (possibly ∞) of a Petrie walk. Edge-stabilizers are always subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

In [6] it was shown that the automorphism group of every edge-transitive map in an orientable surface is one of 14 types determined by the edge-, vertex-, face-, and Petrie walk-stabilizers. Of these 14 types, exactly eight are realizable as infinite maps in the plane, of which only five are realizable by infinitely-ended planar maps. These types are identified by the labels: $2^{\mathrm{P}}, 3, 4, 4^*$, and 4^{P} (cf. [6, Table 2 on p. 19 and Table 4 on p. 35]). Maps corresponding to types 3 and 4 are not vertex-transitive and can be eliminated. Maps corresponding to type 2^{P} have even covalence, and the shortest circuits that separate ends are Petrie circuits ([6, Theorem 5.3]). Since all elements of their cycle spaces are even, these maps are bipartite and therefore imprimitive.

Our candidates for primitivity must therefore correspond to type 4^* or type 4^{P}. In both cases, the lengths of the circuits that separate ends are not generally known. For both of these types, the edge-stabilizers are trivial and they have the same kind of vertex-stabilizers; if $x \in V(\Gamma)$, then its stabilizer $[\mathrm{Aut}(\Gamma)]_x$ is generated by the corner reflections θ_{xf} at alternate corners about x. Hence σ_x^4 is extendable to a map-automorphism of Γ of order $\rho(x)/4$, but σ_x^2 is not extendable. It follows that the valence r is always divisible by 4. For any $x \in V(\Gamma)$, we have $|[\mathrm{Aut}(\Gamma)]_x| = |D_r|/4 = r/2$.

In the case of type 4^*, there are two face-orbits. Faces in one orbit have even covalence c_1 while the other covalence c_2 is unrestricted. The stabilizer of a face f in the first orbit is generated by the corner reflections θ_{xf} at all corners (x,f) and hence admits the counterclockwise face-rotation σ_f^2 of order $c_1/2$ but does not admit σ_f. The stabilizer of a face g in the second orbit is a cyclic group of order c_2 generated by σ_g.

Maps corresponding to type 4^{P} are face-transitive but have two orbits of Petrie walks. Their (constant) covalence c is divisible by 4. The stabilizer of each face f is akin to the vertex-stabilizers; it is generated by the corner reflections at alternate corners about f. Hence $[\mathrm{Aut}(\Gamma)]_f$ includes σ_f^4 but not σ_f^2. If (x,f) is a corner that does not admit a corner-reflection, and if Π denotes the Petrie walk through the two edges incident with that corner, then $\mathrm{Aut}(\Gamma)$ contains the glide reflection γ_Π.

An explanation of the behaviors of these various types and their presentations, including detailed proofs, is to be found in [6].

Proof of Theorem 3.2. Let Γ be a vertex-transitive, edge-transitive, infinitely-ended, 3-connected, planar map with valence r and covalences c_1 and c_2. (Again, possibly $c_1 = c_2 = c$.) From the above discussion we know that $\mathrm{Aut}(\Gamma)$ is of type 4^* or type 4^{P}. Hence Γ has a face-orbit such that all faces f in this orbit admit the corner-reflections θ_{xf} at (at least) alternate corners (x,f). Let us fix c_1 as the covalence of the faces in such an orbit. Thus c_1 is even.

Let (v,f) be a corner of Γ, where f has covalence c_1 and the corner-reflection θ_{vf} is an automorphism of Γ. Let $[x,v]$ and $[x',v]$ be the two edges on this corner. Let us define a graph Φ such that $V(\Phi) = V(\Gamma)$ and $E(\Phi) = \{\{\alpha(x),\alpha(x')\}: \alpha \in \mathrm{Aut}(\Gamma)\}$. Thus Φ is edge-transitive. Since $\mathrm{Aut}(\Gamma) \leqslant \mathrm{Aut}(\Phi)$, Φ is also vertex-transitive.

We claim that when each edge of Φ is inscribed in the interior of the unique face (of Γ) incident with its two endvertices, then Φ is a planar map. For if two edges of Φ were to cross, then they must do so within the interior of a face of Γ. Let a,b,c,d be consecutive vertices incident with a c_1-covalent face $h \in F(\Gamma)$ and suppose that $\{a,c\},\{b,d\} \in E(\Phi)$. Hence there exists some $\alpha \in \mathrm{Aut}(\Gamma)$ such that $\{\alpha(a),\alpha(c)\} = \{b,d\}$. But then $\alpha(h) = h$, and so α is either a nonidentity element of the stabilizer of $[b,c] \in E(\Gamma)$ or $\alpha = \sigma_h^{\pm 1}$. However, no such automorphism belongs to $\mathrm{Aut}(\Gamma)$.

In this regard, we can say more about the subgraph of Φ spanned by the subset of its edges inscribed in any c_1-covalent face $h \in F(\Gamma)$. If $\mathrm{Aut}(\Gamma)$ is of type 4^* and $c_1 \geqslant 8$, then these edges form a circuit, called a *facial circuit*, of length $\frac{1}{2}c_1$ on alternate vertices of the boundary of h. Otherwise we have only an independent set of $\frac{1}{2}c_1$ edges.

Denote by Ψ the component of Φ such that $\{x,x'\} \in E(\Psi)$. Clearly $V(\Psi)$ could have been constructed via Proposition 2.1 starting with $B_0 = \{x,x'\}$ and the group $\mathrm{Aut}(\Gamma)$, from which we conclude that $V(\Psi)$ is indeed a block of imprimitivity of Γ. Thus it remains only to show that $V(\Psi) \neq V(\Gamma)$.

Let Δ be a circuit in Φ. Arbitrarily choose then fix one side of Δ, and with respect to this side, define the *weight* of a vertex $v \in V(\Delta)$ to be the number of vertices on the chosen side of Δ that are adjacent in Γ to v. The *weight* of Δ is the sum of the weights of its vertices. If Δ is a facial circuit, then all of its vertices have weight 0 or all have weight r. If Φ contains any nonfacial circuit, we may consider the collection of all shortest nonfacial circuits and then the subcollection of those with the least weight. Let us denote this subcollection by $\mathscr{S}$, the common length of the circuits in $\mathscr{S}$ by s, and the weight of these circuits by t.

We claim that if $\Delta \in \mathscr{S}$, then all vertices of Δ have the same weight t/s. If this were not so, we could assume without loss of generality the vertices y, x, x', z are consecutive in clockwise order on Δ, that the weight (with respect to Δ) of x exceeds the weight of x', and that the chosen side of Δ is on the right as one traverses Δ in this clockwise sense. Recalling that $\theta_{vf} \in \mathrm{Aut}(\Gamma)$, let $\Delta' = \theta_{vf}(\Delta)$. (In Fig. 3 we have drawn Δ in thick lines and Δ' in thin lines.) Since these two circuits cross at the edge $\{x, x'\}$, they must cross at least once more. Let w be the first vertex of $\Delta \cap \Delta'$ encountered when moving counterclockwise from y along Δ. Let Π_1 denote the subpath of Δ joining x to w and containing y, and let Π_3 be the path through z in Δ joining x to $w' \coloneqq \theta_{vf}(w)$. Denote the subpath of Δ joining w and w' but not containing x by Π_2. For $j = 1, 2, 3$, let $\Pi'_j = \theta_{vf}(\Pi_j)$, and let ℓ_j denote the length of Π_j. Thus $\ell_1 + \ell_2 + \ell_3 + 1 = s$.

Since the circuit $\Pi_1 \cup \Pi'_3$ has length less than s, it must be facial, and the same holds for $\Pi'_1 \cup \Pi_3$. Since no edge of Φ belongs to more than one facial circuit, the following two circuits are nonfacial: $\Delta_1 = \Pi_1 \cup \Pi_2 \cup \Pi'_1$ of length $2\ell_1 + \ell_2 + 1$ and $\Delta_2 = \Pi_3 \cup \Pi_2 \cup \Pi'_3$ of length $2\ell_3 + \ell_2 + 1$. It follows that $\ell_1 = \ell_3$, and so Π_1 and Π_3 must contain nonterminal vertices. Since Δ_1 and Δ_2 have length s, their weight cannot be less than t.

The weight of Δ_1 may be computed from the weight of Δ by replacing the contributions from the nonterminal vertices on Π_3 by the contributions from the nonterminal vertices on Π'_1; each the former contributes r to the weight of Δ, whereas their replacements in Π'_1 contribute 0 the weight of Δ_1. Meanwhile, the weights of x' and w' with respect to Δ_1 are the same as their weights with respect to Δ. Thus the weight of Δ_1 is less than t, giving a contradiction and proving our claim.

We make two observations:

Observation 1. If $\{u, z\}$ and $\{u, z'\}$ are consecutive edges of Ψ around some vertex $u \in V(\Psi)$, then there exists a face $g \in F(\Gamma)$ such that $\theta_{ug}(z) = z'$.

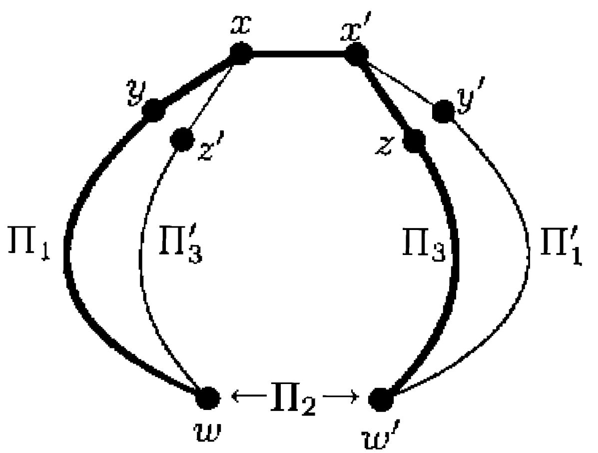

Fig. 3.

Observation 2. If $\Delta \in \mathscr{S}$ is nonfacial, then adjacent edges of Δ are contained in distinct faces of Γ.

To justify this second observation, note that if two adjacent edges of Δ were inscribed in the same face of Γ, then the weight in Δ of their common incident vertex would be 0 or r. Hence all of the vertices of Δ would have weight 0 or all would have weight r, in which case Δ would be a facial circuit.

We next claim that if Ψ contains a nonfacial circuit Δ, then $V(\Psi) \neq V(\Gamma)$. We will show that vertices lying within the chosen side of Δ cannot be vertices of Ψ. Since Ψ is connected, there exists an edge $\{u', u\} \in E(\Psi)$ such that $u \in V(\Delta)$ and u' lies within the chosen side of Δ.

Let z be a vertex on Δ adjacent to u and select z' on the chosen side of Δ so that $\{z, u\}$ and $\{z', u\} \in E(\Psi)$ are consecutive around u (in Ψ) and the "weight" of u between these two edges is less than $\frac{t}{s}$. By Observation 1, there exists a face g such that $\theta_{ug}(z) = z'$. Let $\Delta' = \theta_{ug}(\Delta)$ and similarly define w, w', Π_j, Π'_j, etc., as above and as shown in Fig. 4(a). Since each of the paths Π_1, Π'_1, Π_3, and Π'_3 contains two consecutive edges from Δ or Δ', Observation 2 implies that no circuit containing any of these paths is a facial circuit. Then one easily sees that $w' = w$, and so all four of these paths have the same length. Hence the circuit $\Pi'_3 \cup \Pi_3$ has length s, but its weight is less than t because all of its vertices except u and w have weight $\frac{t}{s}$ while w has weight at most $\frac{t}{s}$ and u has weight less than $\frac{t}{s}$, giving a contradiction and proving the claim.

Let x, x', v, and f be as defined at the outset of this proof and depicted in Fig. 4(b). (Edges of Γ are displayed in the figure as double lines.) For the final argument of this proof, we show that if $v \in V(\Psi)$, then Ψ must contain a nonfacial circuit; by the previous claim, this will be sufficient to conclude that Γ is imprimitive.

Since Ψ is connected, it contains an xv-path Σ. The subgraph $\Sigma \cup \theta_{ug}(\Sigma)$ either is a circuit or contains various circuits formed by subpaths of Σ and $\theta_{ug}(\Sigma)$ between crossings. If any such circuit is nonfacial, we are done. So assume that all such circuits are facial. As one moves along Σ from x toward v, let w be the first vertex of $\Sigma \cap \theta_{ug}(\Sigma)$ encountered. (Possibly $w = v$.) Let $\Delta := \Sigma[x, w] \cup \theta_{ug}(\Sigma[x, w]) \cup \{x, x'\}$. Since Δ is a facial circuit, all facial circuits in Ψ must have odd length. Since Δ is inscribed in the face f, we also have $w \neq v$.

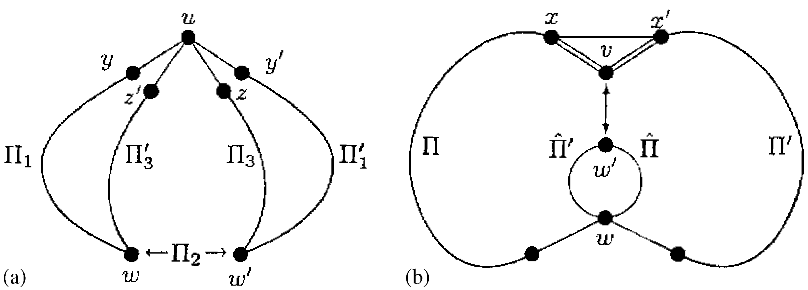

Fig. 4.

The only way that v could be incident with an edge of Ψ that is fixed by θ_{xf} would be if $c_1 = 4$, but in that case, as noted above, Φ contains no facial circuits. Hence there exists a pair of vertices w' and w'' on $\Sigma[w,v] \cap \theta_{ug}(\Sigma[w,v])$ such that $\Sigma[w',w''] \cup \theta_{ug}(\Sigma[w',w''])$ is a circuit and hence a facial circuit. However, its length is even, giving a contradiction. Thus $v \notin V(\Psi)$ and the proof of the theorem is complete. □

Corollary 4.4. *If Γ is an edge-transitive, 3-connected, planar map such that* Aut(Γ) *is of type* 4, 4^*, *or* 4^{P}, *then the union of all the vertex- and face-stabilizers of Γ does not generate* Aut(Γ).

Proof. First suppose that Aut(Γ) is of type 4^*, or 4^{P}. Then Γ is vertex-transitive. Let H denote the union of all the vertex- and face-stabilizers of Γ, and consider the graph Ψ in the proof of Theorem 3.2. One verifies that every automorphism in H maps Ψ onto itself. But since $V(\Psi)$ is a proper subset of $V(\Gamma)$, H does not generate any vertex-transitive subgroup of Aut(Γ).

Since maps whose automorphism groups are of type 4 are exactly the planar duals of maps whose automorphism groups are of type 4^*, and since the set H is preserved under planar duality, the corollary also holds in the case of type 4. □

5. Forbidden combinations of layers

In this section, we prove Lemmas 3.4 and 3.5. The first proof cites the following.

Proposition 5.1 (Tutte [20]). *If a finite graph is vertex-transitive, edge-transitive, and odd-valent, then it is arc-transitive.*

Lemma 3.4. *Let Γ be an infinite, primitive graph that is not edge-transitive. If one of its layers has connectivity exactly 1 with lobes that are isomorphs of K_4, then Γ* is *not planar.*

Proof. Suppose that Γ is an infinite, primitive graph, and let Θ_1 and Θ_2 be two layers of Γ. Suppose that $\kappa(\Theta_1) = 1$ and that all its lobes are all congruent to K_4. Let Λ with $V(\Lambda) = \{x_0, x_1, x_2, x_3\}$ be a lobe of Θ_1.

Suppose that Γ is planar and, without loss of generality, that when Γ is embedded in the plane, then x_0 lies in the interior of the circuit $\langle x_1, x_2, x_3 \rangle$.

Observe that for $\alpha \in \mathrm{Aut}(\Gamma)$, one has $\alpha(\Lambda) = \Lambda$ if and only if $E(\Lambda) \cap \alpha(E(\Lambda)) \neq \emptyset$. Let $H := \{\alpha \in \mathrm{Aut}(\Gamma) : \alpha(\Lambda) = \Lambda\}$. The restriction $H_{|\Lambda}$ of H to Λ acts vertex-transitively on Λ. It also acts edge-transitively on Λ, for if it acted only cyclically on $V(\Lambda)$, then it would act imprimitively. Since Λ is odd-valent, $H_{|\Lambda}$ acts arc-transitively on $|\Lambda$ by Proposition 5.1 and hence transitively on the set of ordered pairs of distinct vertices of Λ. Thus $H_{|\Lambda}$ contains or equals the alternating group on the four vertices of Λ. In particular, H includes automorphisms $\zeta = (x_0, x_2)(x_1, x_3)$ and $\eta = (x_0)(x_1, x_2, x_3)$. Note that η fixes the interior of the circuit $\langle x_1, x_2, x_3 \rangle$ while ζ reverses orientation by interchanging its interior and exterior.

By Proposition 3.3, Θ_2 contains a path joining x_0 to this circuit. Without loss of generality, we may suppose that this is an x_0x_1-path, and we denote it by Σ_{01}. Except for its end-vertices, Σ_{01} lies either entirely in the interior of $\langle x_0, x_1, x_2 \rangle$ or entirely in the interior of $\langle x_0, x_1, x_3 \rangle$, and without loss of generality, we opt for the latter circuit.

We define $\Sigma_{0j} := \eta^{j-1}(\Sigma_{01})$ for $j = 1, 2, 3$. We also define $\Sigma_{12} := \eta^{-1}\zeta(\Sigma_{01})$, $\Sigma_{13} := \eta\zeta(\Sigma_{01})$, and $\Sigma_{23} := \zeta(\Sigma_{01})$. Except for their end-vertices, Σ_{02} and Σ_{03} are contained in the interiors of $\langle x_0, x_1, x_2 \rangle$ and $\langle x_0, x_2, x_3 \rangle$, respectively, while both Σ_{13} and Σ_{23} are in the exterior of $\langle x_1, x_2, x_3 \rangle$ [see Fig. 5]. We see that $\Lambda' := \Sigma_{01} \cup \Sigma_{02} \cup \Sigma_{03} \cup \Sigma_{12} \cup \Sigma_{13} \cup \Sigma_{23}$ is a subgraph of Θ_2 that is homeomorphic to K_4. Hence $\Lambda' \cong K_4$, and $V(\Lambda') = V(\Lambda)$. But then Σ_{01} has only one edge $[x_0, x_1]$, and this edge belongs to Θ_1, giving a contradiction. □

Lemma 3.5. *Let Γ be a vertex-transitive graph that is not edge-transitive. Suppose that two of the layers of Γ are primitive subgraphs whose lobes are circuits of odd prime length. Then Γ is not planar.*

Proof. Suppose that Γ is an infinite, vertex-transitive, planar graph that is not edge-transitive. For $i = 1, 2$, let Θ_i be a layer of Γ such that $\kappa(\Theta_i) = 1$ whose lobes are circuits of length p_i, and let Δ_i be a lobe of Θ_i.

Suppose that $\Delta_1 \cap \Delta_2$ contains at least two vertices. (Clearly this intersection contains no edge.) By planarity there exist $x, y \in V(\Delta_1 \cap \Delta_2)$ that are simultaneously consecutive vertices of this intersection with respect to both circuits. Since Θ_i is vertex- and edge-transitive and p_i is an odd prime, there exists an orientation-preserving automorphism $\sigma \in \mathrm{Aut}(\Gamma)$ that rotates Δ_1 by exactly one vertex (and by one edge). Since the distinct lobes Δ_2 and $\Delta_2' = \sigma(\Delta_2)$ of Θ_2 have at most one vertex

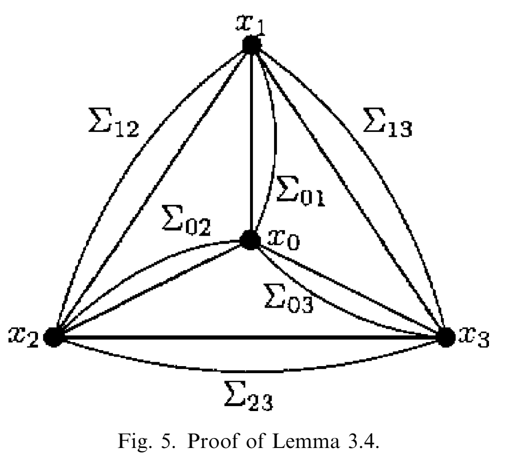

Fig. 5. Proof of Lemma 3.4.

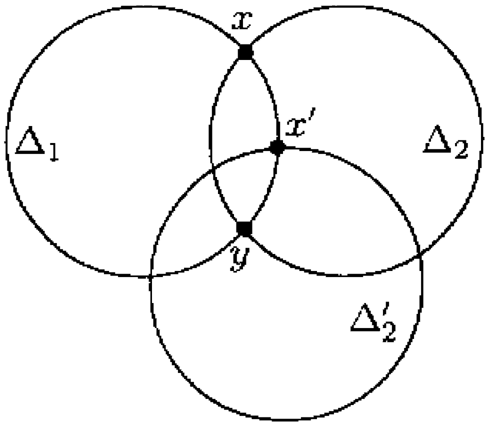

Fig. 6. Proof of Lemma 3.5.

in common and hence cannot cross, and since they lie on the same side of Δ_1, we have that x and y are adjacent on Δ_1 see Fig. 6. By a symmetrical argument, we simultaneously have $[x, y] \in E(\Delta_2)$, which is impossible. We've shown that lobes from distinct layers have at most one vertex in common.

Suppose that $V(\Delta_1) = \{x_0, x_1, \ldots, x_{p_1-1}\}$ where the vertices are listed in cyclic order. By Proposition 3.3, Θ_2 is connected and hence contains a shortest path Σ_0 from x_0 to x_1. Let $\Sigma_j = \sigma^j(\Sigma_0)$ for $j = 2, \ldots, p_1 - 1$. Note that Σ_0 followed by Σ_1 is a trail in Θ_2 from x_0 to x_2 and hence contains a path from x_0 to x_2; let y_1 be the vertex on this path common to Σ_0 and Σ_1. Defining y_j for $j = 2, \ldots, p_1 - 1, p_1 \equiv 0$ in the same manner, we conclude by the arrangement of lobes in Θ_2 that either $z \coloneqq y_0 = y_1 = \cdots = y_{p-1}$ or these vertices y_j lie on a circuit Z in Θ_2 whose length is a multiple of p_1. In the latter case, $p_2 = p_1$ and this circuit is a lobe of Θ_2. We note that, since the only circuits in Θ_2 are entire lobes, no two paths in Θ_2 emanating from a lobe of Θ_2 and originating at different vertices on that lobe may ever meet.

The first edge on the path in Θ_2 from x_0 to the central vertex z or circuit Z must belong to a lobe Ω of Θ_2; let σ_2 denote a clockwise rotation by one edge around Ω. (In Fig. 6 we have drawn in the images $\Delta' = \sigma_2(\Delta_1)$ and $z' = \sigma_2(z)$.) The image under σ_2 of the path in Θ_2 joining z to x_1 must be a path joining z' to $\sigma_2(x_1)$ on $\sigma_2(\Delta_1)$ and it must also be disjoint from Ω. As we have noted, no path from z' other than the one joining it to Ω can meet Ω or the path from Ω to x_{p-1} via z, yet by the planarity of Γ, this is precisely what such a path must do. $\square$

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