# MAKING CHANGE EFFICIENTLY

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Suppose U.S. coinage were to be drastically reformed so that coins of only *one* denomination beyond the penny were minted. What value, $k$, of this coin would allow change to be made most efficiently? More generally consider a system of currency where the smallest paper note has value $d$¢. Making change then means being able to hand over, in 1¢ and $k$¢ coins, any value from 1¢ to $(d-1)$¢. What value for $k$ makes this system most efficient?

Even this two-denomination problem is not so simple. The difficulty lies in the fact that the solution requires a minimum of a maximum of a minimum. In this paper, we describe how to lead a student to solve completely the two-denomination problem for any currency, while developing the skills needed to tackle the much harder three-denomination problem. While calculus is not needed for the two-denomination problem, it proves to be useful for the extension to three denominations. In any case, someone tackling this project needs the same level of comfort with functions and graphs as a calculus student. It is hoped such a student will find these exercises enjoyable as well as enlightening.

## Warm-up Problems

**Problem 1.** For current US coinage (1¢, 5¢, 10¢, 25¢ and 50¢ coins), what amount of change requires the most coins? How many coins is that? What is the least number of coins that you must carry so that you can make *any* change of 99¢ or less? Which coins do you need?

**Problem 2.** Again consider US coinage but now suppose that coins are minted in only *one* denomination beyond 1¢. Let $k$ denote this denomination and answer the following questions for $k$ equal to 5, 10, 25 and 50.

(i) What amount (or amounts) of change requires the maximum number of coins? How many coins is that?

(ii) What is the least number of coins that you must carry so that you can make any amount of change? Which coins are required?

On the basis of your solution to this problem, if you were to mint only coins in two denominations, 1¢ and $k$¢, which value of $k$ in the above range would you choose and why?

## Measuring Efficiency

There are two slightly different measures of efficiency in the coinage problem. We can ask for the $k$ that minimizes the maximum number of coins that some *particular* amount of change demands. Or we can ask for the $k$ that minimizes the number of coins that need to be carried so as to make *any* amount of change. The following problems make each of these precise and show that they are only slightly different.

First fix the denomination $k$ and consider the function:

$f_k(t)$ = the minimum number of coins needed to make $t$¢ in change.

Figure 1 displays the saw-tooth shaped graph of $f_k(t)$. We are actually interested in the maximum of $f_k(t)$ over all $t$ less than $d$, . This leads us to define

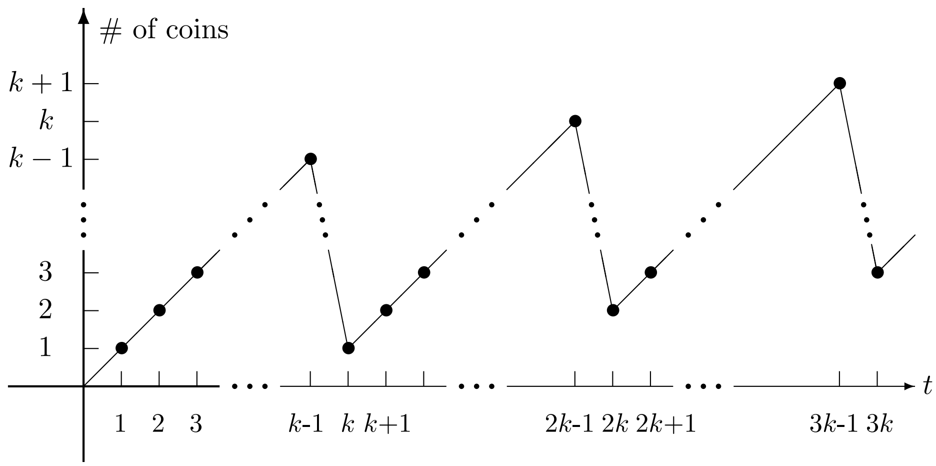


FIGURE 1

$$g_k(t) = \max\{f_k(t) \mid 1 \le t \le d-1\}.$$

A nice feature of $g_k(t)$ is that it is the floor of a piece-wise linear function.

**Problem 3.** Show that $g_k(t) = \lfloor G_k(t) \rfloor$ where

$$G_k(t) = \begin{cases} t & \text{if } t \le k-1 \\ \frac{1}{k}t + \frac{(k-1)^2}{k} & \text{if } t \ge k-1 \end{cases}$$

(*Hint:* Figure 2 shows the graphs of $g_k(t)$ and $G_k(t)$ together. Find the

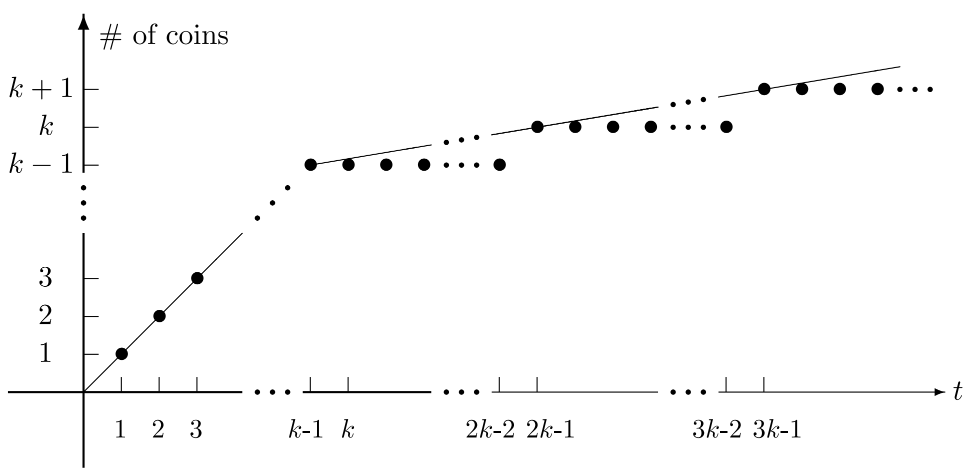


FIGURE 2

slope of the second segment of $G_k(t)$ and show that it passes through the point $(k-1,k-1)$.)

**Problem 4.** Let $h_k(t)$ denote the minimum number of 1¢ and $k$¢ coins one must carry to be able to make *any* transaction of value $t$ or less.

(i) Show that $h_k(t)$ is the floor of a piece-wise linear function $H_k(t)$ and find a formula for $H_k$.
(ii) Show that, for $t \geq k$, $H_k(t) = G_k(t) + (1 - \frac{1}{k})$.

Problems 3 and 4 show that our two measures of efficiency are really the same. We choose the functions $g_k(t)$ and $G_k(t)$ with which to continue our investigation since $G_k(t)$ is easier to use.

## Pursuing Efficiency

For each $t$, we want to find the minimum of $G_k(t)$ over all values of $k$. This leads us to define still another function: $R(t) = \min\{G_k(t) \mid k > 0\}$.

**Problem 5.**

(i) Graph $G_1(t)$, $G_2(t)$, $G_3(t)$, $\cdots$, $G_6(t)$ on the same axes.
(ii) Graph $R(t)$ for $0 \leq t \leq 41$. Note that $R(t)$ is piecewise, determined by the $G_k(t)$. Check your answer against Figure 3.
(iii) Show that

$$R(t) = \tfrac{1}{k}t + \tfrac{(k-1)^2}{k}$$

for $(k^2 - k - 1) \leq t \leq (k^2 + k - 1)$, $k = 1, 2, \cdots$

Is there a smooth function through the intersection points defining $R(t)$? It turns out there is. The first step towards finding it is to represent these intersection points parametrically. In Figure 3 the points are $(1, 1)$, $(5, 3)$, $(11, 5)$, $(19, 7)$, $(29, 9)$ and $(41, 11)$.

**Problem 6.**

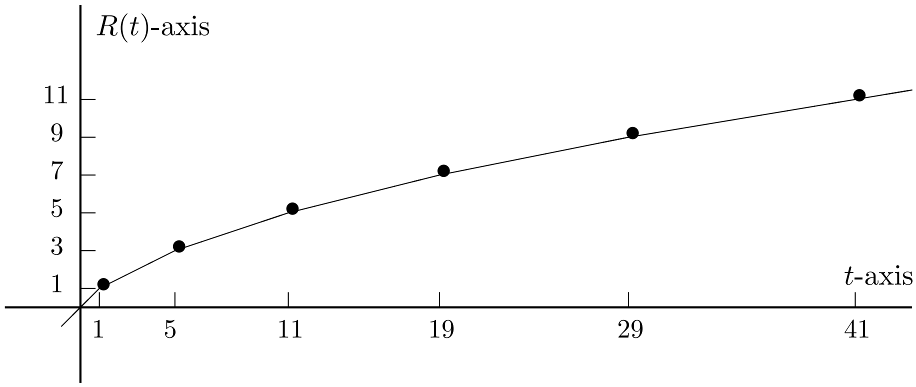


FIGURE 3

(i) Let $(x, y)$ denote the left hand endpoint of the segment of the graph of $R(t)$ defined by $G_k$; verify that

$$x = k^2 - k - 1$$

and

$$y = \frac{1}{k}(k^2 - k - 1) + \frac{(k-1)^2}{k} = 2k - 3.$$

(ii) Show that $x = \frac{y^2+4y-1}{4}$, and $y = \sqrt{4x+5} - 2$.

Thus, we may replace $R(t)$ by $S(t) = \sqrt{4t+5} - 2$. If $n = \lfloor S(t) \rfloor$, then $n$ is the maximum number of 1¢ and $k$¢ coins needed to make any transaction of value $t$ or less, *if* $k$ is selected optimally. Now not only would we like to know the least number of coins needed to make any transaction of value $t$ or less, but we would also like to know the denomination $k$ that gives this minimum: $K(t)$ is defined to be the integer $k$ so that $R(t) = G_k(t)$.

How do we select $k$ properly? In part (i) of Problem 6, we showed that $n = y = 2k-3$; so, $k = \frac{n+3}{2} = \frac{\lfloor S(t) \rfloor + 3}{2}$. Actually, when $n = 2k-3$, we are at the left hand end point of the segment of $R(t)$ defined by $k$, which is also the right hand end point of the interval defined by $k-1$. Thus, when $n$ is odd, we may take $k$ to be either $\frac{n+3}{2}$ or $\frac{n+1}{2}$. When $n$ is even, one easily checks that $n$ corresponds to the $y$-coordinate of the center of the segment of $R(t)$ defined by $k$ where $n = 2k-2$. Hence we have

$$K(t) = \lfloor \frac{n+2}{2} \rfloor = \lfloor \frac{\lfloor S(t) \rfloor + 2}{2} \rfloor = \lfloor \frac{S(t)+2}{2} \rfloor = \lfloor \frac{\sqrt{4t+5}}{2} \rfloor,$$

with the understanding that $k = K(t)$ is often one of *several* choices for $k$.

## Examples

**Problem 7.** Show that making change in U.S currency can be carried out with 18 or fewer 1¢ and $k$¢ coins whenever $k$ is in the range: $8 \leq k \leq 13$.

If the value of the dollar were altered slightly, to 99¢, the result is dramatically different.

**Problem 8.** Show that change of 98¢ or less may be made with 17 or fewer 1¢ and $k$¢ coins, but only when $k = 10$.

**Problem 9.** In Great Britain, before 1971, the pound was divided into 20 shillings each of which was worth 12 pence. In this system, the smallest bill was one pound, which was thus worth 240 pence. What is the most efficient two-coin system under these circumstances? Is the ideal second coin the shilling?

**Problem 10.** After 1971, Britain adopted a decimal system which is still in use. A pound is worth 100 (new) pence. However, the smallest bill is the five pound note: worth 500 pence. What is the most efficient two-coin system now?

## Another Approach

There are usually several ways to solve a problem. As we have already seen, we could use a second, slightly different measure of efficiency. Another possibility is to use $n$ as our independent variable instead of $t$. In this case, we consider $m_k(n)$ defined to be the maximum $t$ so that all transactions of value $t$ or less can be made using at most $n$, 1 and $k$ cent coins. The function $m_k(n)$ is the inverse of the function $g_k(t)$. It is not too difficult to compute.

**Problem 11.**

(i) Show that

$$m_k(n) = \begin{cases} n & \text{if } n < k \\ kn - (k-1)(k-2) & \text{if } n \geq k \end{cases}$$

(ii) Verify that $m_{10}(17) = 98$ and explain how this was predicted by the previous treatment.

We may think of $m_k(n)$ as a function of either $n$ or $k$. Fixing $n$, we may ask for the value of $k$ which maximizes this function. As a function in $k$, $m_k(n)$ is quadratic:

$$m_k(n) = -k^2 + (n+3)k - 2.$$

Differentiating, we see that $m_k(n)$ is maximized, as a function of $k$, at $\frac{n+3}{2}$. Thus, for a given value of $n$, the most efficient choice of $k$ is $\frac{n+3}{2}$ or $n = 2k - 3$. This leads us to define the *perfect 2-coin currencies* as those with $n = 2k - 3$ and dollar $d$ amount given by $d = m_k(n) + 1 = m_k(2k-3) + 1 = k^2 - 1$.

**Problem 12.** Verify that $m_k(2k-3) + 1 = k^2 - 1$.

Table 1 lists the first few *perfect 2-coin currencies*:

| $k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|
| $n = 2k-3$ | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 |
| $d = k^2 - 1$ | 3 | 8 | 15 | 24 | 35 | 48 | 63 | 80 | 99 | 120 |

Table 1

In retrospect, the approach using $m(n)$ turns out to be more efficient than using $g(t)$. If this problem were considered important enough to appear in textbooks, it would probably be presented that way. Much of mathematics is presented in texts in refined and often slick versions which do not represent the path of first discovery. This can be very misleading. It gives the impression that one must first understand the *right way* to attack a problem and then solve it. As a result, many students spend their time trying to identify this right way and, being unable to do so, give up.

A better approach to a mathematical problem is very much like writing a paper. First of all, get something down, even something that you will eventually discard! In the case of mathematics, list what you know and work a few examples. Then follow any leads you see. A good writer also periodically reviews what is written, then edits and perhaps reorganizes it. Exactly the same approach works here. In both writing and solving a mathematical problem, one proceeds by cycling among different tasks forging ahead with a particular line of reasoning, reflecting on what is produced, reorganizing that material, and finally identifying a new direction for study. In good writing and in good mathematics, one must be prepared to discard much of the work one has done.

## The 3-denomination Problem

The next stage is to consider the three coin problem. Assume that $d$ is the denomination of the smallest bill, select positive integers $k$ and $h$ $(1 < k < h)$ so that the maximum number of 1¢, $k$¢, and $h$¢ coins needed to make change for all values of $d - 1$ or less is as small as is possible. There are two cautions that I would give to any student advancing to consider 3-denominations: be willing to spend a good bit of time on this problem, and understand that you may not be able to solve the problem completely!

**Project I.** A natural hypothesis is that $h$ should be chosen as the solution to the 2-denomination problem with dollar amount $d$ and that $k$ should be chosen recursively as the solution to the 2-denomination problem with dollar amount $h$. Try it?

**Project II.** Another route worth investigating is to imitate our first approach to the two coin problem. Specifically, compute the function $g_{k,h}(t)$ defined to be the maximum of the numbers of 1¢, $k$¢ and $h$¢ coins needed to make each transaction from 1 to $t$.

It turns out, the computation of $g_{k,h}(t)$ can be quite messy. Instead write a computer program for $g_{k,h}(t)$ inductively from 1 to $t$. Then one can easily make guesses and check them.

**Project III.** At a certain point in any research problem one must find out what has already been done. Just when it is best to do this is not clear. Reading what others have done at the start may save you a lot of work; on-the-other-hand, it may lead your thinking away from directions you might otherwise take – directions that may prove to be quite productive.

This coinage problem is particularly hard to research since references to it are widely scattered in the literature. A few starting points are [**1**, p191], [**2**], [**3**], and [**4**, pp 207-217]

## References

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