# RESEARCH QUESTIONS FROM ELEMENTARY CALCULUS

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Consider the following typical optimization problem from elementary calculus. A farmer wishes to construct two identical rectangular enclosures by dividing a single enclosure down the center. If the farmer has 120 feet of fencing, what are the dimensions of the enclosures of maximum area that can be constructed? One easily checks that the optimal enclosures are pictured below.

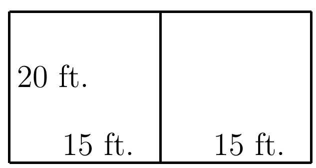



FIGURE 1.

Each enclosure of this optimal solution has area of $15 \times 20 = 300$ square feet. If the farmer wishes to construct two identical rectangular, 300 square foot enclosures side by side while minimizing the length of fencing used, the optimal configuration is the same. This represents a very general, but hardly new, observation about constrained optimization: reversing the roles of the constraint and objective functions frequently results in a problem with the "same" solution. Well known or not, it should be proved. And here, as in much research, finding the right level of generality and then properly formulating the result are integral parts, perhaps the most important parts, of the research. So, the first research problem in this research project is to formulate and prove this "dual optimization" principle. We continue our discussion assuming that an appropriate version of the dual optimization principle has been proved and simply refer to a configuration like the one pictured above as the optimal configuration without specifying which of the two optimization problems has been posed.

Referring to the above optimal configuration, we make a second observation: the amount of east-west fencing equals the amount of north-south fencing. This is true of the solutions to many such optimization problems. Consider the above problem but suppose that less expensive fencing can be used to divide the larger rectangle and that minimizing the cost to enclose a fixed area (or maximizing the area enclosed for a fixed cost) is the object. In this case, the optimal solution will be such that the *cost* of the east-west fencing equals the *cost* of the north-south fencing. For another example, consider the problem of constructing a rectangular enclosure along the side of a barn. See Figure 2. Assuming the size of the enclosure is small relative to the side of the barn, the optimal solution will have half the fencing parallel to the side of the barn and half perpendicular. So we see that this "half and half" principle is valid in a variety of settings. Again, finding the right level of generality and properly formulating the result are essential parts of the project.

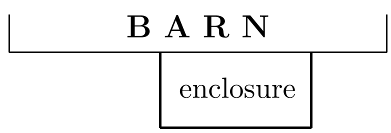



FIGURE 2.

Both of these principles have three dimensional analogues which can be investigated. For example, consider building a chest using inexpensive wood for the bottom and back, moderately expensive wood for the sides and front and expensive inlay for the top. Then maximizing volume for a fixed cost or minimizing cost for a fixed volume will yield the same optimal configuration (i.e. the same ratios between width, depth and height) and that configuration will be the one in which the total cost for the top and bottom equals the total cost of the two sides which, in turn, equals the total cost of the front and back.

Later in this note we outline a proof of a limited formulation of these two principles. However they are valid in rather general settings and the first of the research projects we set forth is to state and prove these results at an appropriate level of generality:

**Research Project 1.** *State and prove a general formulation of the dual optimization principle and of the half and half principle.*

We note that, once these two results have been stated and proved, virtually all standard enclosure and box problems can be solved by a few simple algebraic steps.

Returning to two dimensions, we pursue another line of inquiry. Reconsider the problem of constructing two identical rectangular enclosures. If the basic structure is not constrained by the condition that it be constructed by dividing a single enclosure down the center, the solution remains the same but the problem is somewhat harder. One must first prove that an optimal solution occurs only when the two enclosures share a common side. The problem can be made even more interesting by requiring only that the enclosures be rectangular and have the same area. Again, by dropping the condition that the rectangles be congruent, the problem becomes a bit harder but the solution remains the same.

Increasing the number of rectangles leads to two very interesting collections of problems.

**Problem $\mathbf{C_n}$:** Find the optimal configuration for $n$ congruent rectangular enclosures.

**Problem $\mathbf{A_n}$:** Find the optimal configuration for $n$ rectangular enclosures of equal area.

**Research Project 2.** *Investigate the solutions to the problems* $\mathbf{C_n}$ *and* $\mathbf{A_n}$*, for all* $n$*.*

To give the reader a taste of this project, we sketch the solutions to $\mathbf{C_n}$ and $\mathbf{A_n}$, for $n = 2$ and $n = 3$. Investigating these problems will be greatly facilitated by the two principles described in Research Project 1 and we start this investigation by proving somewhat limited formulations of these principles. Suppose a configuration of $n$, $x \times y$ rectangles all with the same orientation has been selected. Problem $\mathbf{C_n}$ for this configuration can be formulated as:

$$\text{Maximize } a = xy \text{ subject to } rx + sy = f, \text{ or}$$

$$\text{Minimize } f = rx + sy \text{ subject to } xy = a,$$

where $x$ and $y$ are the dimensions of the congruent rectangular enclosures, $r$ and $s$ are constants which depend on the configuration, $f$ is a fixed length of fencing and $a$ the area or $a$ is a fixed area and $f$ is the length of fencing needed. In either case, the constraint equation permits us to think of $y$ as a function of $x$. If we differentiate both equations with respect to $x$, we get:

$$\tfrac{df}{dx} = r + s\tfrac{dy}{dx} \text{ and } \tfrac{da}{dx} = y + x\tfrac{dy}{dx}.$$

Maximizing $a$ for fixed $f$, we have $\frac{df}{dx} = 0$ and set $\frac{da}{dx} = 0$; minimizing $f$ for fixed $a$, we have $\frac{da}{dx} = 0$ and set $\frac{df}{dx} = 0$. In either case, $x$ and $y$ must satisfy the system:

$$r + s\tfrac{dy}{dx} = 0 \text{ and } y + x\tfrac{dy}{dx} = 0.$$

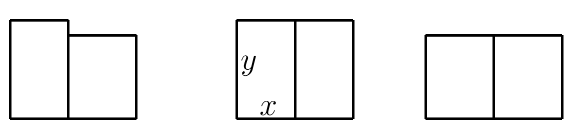
From this we conclude that, for each problem, the optimal configuration satisfies $rx = sy$. Thus, we have verified both principles for a special case of the problem $\mathbf{C_n}$, the case when all enclosures have the same orientation. Actually, the half and half principle need not hold for the general $\mathbf{C_n}$ problem but the dual optimization does hold for the general $\mathbf{C_n}$ problem. Assuming the dual optimization principle, we can restrict our investigation of problem $\mathbf{C_n}$ to a standard form: Minimize the total amount of fencing needed to construct $n$ congruent rectangular enclosures each with an area of one square unit. For each possible configuration, we must

$$\text{Minimize } rx + sy \text{ subject to } xy = 1$$

and then select the configuration with the smallest minimum. By the half and half principle, the minimum will occur when $rx = sy$. Thus $rx = s\frac{1}{x}$, $x = \frac{\sqrt{s}}{\sqrt{r}}$, $y = \frac{\sqrt{r}}{\sqrt{s}}$ and $f = rx + sy$ has a minimum of $2\sqrt{rs}$. Armed with this information, the problem becomes a combinatorial/geometric problem of finding the configuration for which $2\sqrt{rs}$ has its smallest value.

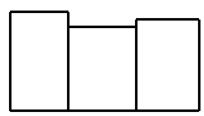
The discernible reader will have noted that we have made some basic assumptions about the optimal solutions to $\mathbf{C_n}$. First, we have assumed that all optimal configurations for $\mathbf{C_n}$ satisfy the condition that the sides of all rectangles are parallel to one of the axes of a fixed pair of orthogonal axes. Early in any investigation one would want to prove this for both $\mathbf{C_n}$ and $\mathbf{A_n}$. Second, it is easy to see that the constraint function has the form $rx + sy$ when at least one optimal configuration for $\mathbf{C_n}$ has all of its rectangles oriented in the same direction. One must show that the constraint function has this form even if some of the rectangles are "on their sides."

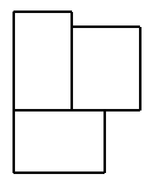
Moving on and assuming that our two principles hold for problem $\mathbf{A_n}$, we reformulate problem $\mathbf{A_2}$ in the standard format: minimize the fencing needed to enclose two rectangles of area 1 square unit each. Clearly, in the optimal configuration, the two rectangles will share one complete side of one of the rectangles:



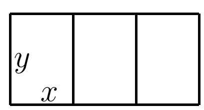

The left hand figure represents the general configuration and the right hand figures represent the two symmetric configurations constructed by reflecting each of the rectangles, in turn, about the common side. One easily checks that the average of the amount of fencing used in the two right hand figures is less than the amount of fencing used in the left hand figure. Thus, we may conclude that the optimal configuration will be symmetric about a common side and that the solution to problem $\mathbf{A_2}$ is the solution to problem $\mathbf{C_2}$. Labeling the central figure, we have that the amount of fencing is $f = 4x + 3y$; so $y = \frac{2}{\sqrt{3}}$, $x = \frac{\sqrt{3}}{2}$, and $f = 4\sqrt{3}$.

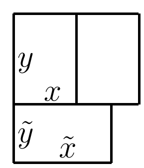
We note that, in general, if an optimal configuration for $\mathbf{A_n}$ consists of congruent rectangles, then it is also an optimal configuration for $\mathbf{C_n}$. Turning to problem $\mathbf{A_3}$, there are two basic configurations, three in a row or three in a cluster:





Using a sequence of symmetry arguments on pairs of enclosures, we can show that the three or two enclosures in a row must be congruent:




In the three in a row case, we have $f = 4y + 6x$ and conclude that the optimal dimensions for this configuration are $y = \frac{\sqrt{6}}{2} = \frac{\sqrt{3}}{\sqrt{2}}$, $x = \frac{2}{\sqrt{6}} = \frac{\sqrt{2}}{\sqrt{3}}$ and $f = 2\sqrt{24} = 4\sqrt{6}$. Turning to the cluster configuration, we first minimize the amount of fencing needed for the two identical enclosures. But that is simply the solution to $\mathbf{A_2}$: $y = \frac{2}{\sqrt{3}}$, $x = \frac{\sqrt{3}}{2}$ and $f = 4\sqrt{3}$. We now treat the third enclosure as if it were "built on the side of a barn." We have $\tilde{f} = \tilde{x} + 2\tilde{y}$; so $\tilde{y} = \frac{1}{\sqrt{2}}$, $\tilde{x} = \sqrt{2}$, and $\tilde{f} = 2\sqrt{2}$. So the total fencing needed for the right hand configuration is $f + \tilde{f} = 4\sqrt{3} + 2\sqrt{2}$, which is slightly less than $4\sqrt{6}$ (9.757 vs. 9.798).

The left hand configuration is a solution to problem $\mathbf{C_3}$ while the right hand configuration (the optimal solution to $\mathbf{A_3}$) is not. Before we can conclude that the left hand configuration is the optimal solution to problem $\mathbf{C_3}$, must consider the cluster configuration with the additional condition that all three rectangles are congruent. This leads to two cases: $\tilde{x} = x$ and $\tilde{y} = y$ or $\tilde{x} = y$ and $\tilde{y} = x$. In the first case, the total amount of fencing is $f = 5x + 5y$ and the optimal solution for this configuration has $x = y = 1$ and $f = 10$. In the second case, the total amount of fencing is $f = 6x + 4y$ and the optimal solution for this configuration is the same as the three in a row configuration. Thus, there are many optimal configurations for problem $\mathbf{C_3}$, all of which involve three $\frac{\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{3}}$ rectangles. In one case, they are in a row and, in the others, one end rectangle is removed and repositioned to form a cluster.

Moving on to $n = 4$, it is not surprising that problems $\mathbf{C_4}$ and $\mathbf{A_4}$ have a common solution: that solution being the $2 \times 2$ grid of unit squares. In fact, a natural conjecture is that, when $n = m^2$, problems $\mathbf{C_n}$ and $\mathbf{A_n}$ have a common solution in the $m \times m$ grid of unit squares. Proving this result might be a good place to start an attack on the general $\mathbf{C_n}$ and $\mathbf{A_n}$ problems.

There is one last natural extension of the $\mathbf{A_n}$ we wish to consider. Let the real numbers $r_1 \geq r_2 \geq \cdots \geq r_n > 0$ be given such that $r_1 + r_2 + \cdots + r_n = 1$. Maximize the total area $a$ that can be divided into $n$ rectangular regions of areas $r_i a$, for $i = 1, \cdots n$, by a fixed amount of fencing. Or, given a fixed area $a$, minimize the amount of fencing needed to enclose $n$ rectangular regions of areas $r_i a$, for $i = 1, \cdots n$. Assuming that the dual optimization principle holds in this case, we denote this last formulation with $a = 1$ as the $\mathbf{R_n}$ problem. To illustrate this class of problems, we consider $\mathbf{R_2}$.

To facilitate our discussion of $\mathbf{R_2}$, we prove a variation on the "half and half" principle. Suppose we wish to minimize $f = a_1 x_1 + b_1 y_1 + a_2 x_2 + b_2 y_2$, where $x_i y_i = r_i$. Completing the square, we see that $a_i x_i + b_i y_i = (\sqrt{a_i x_i} - \sqrt{b_i y_i})^2 + \sqrt{a_i b_i r_i}$ and

$$f = (\sqrt{a_1 x_1} - \sqrt{b_1 y_1})^2 + (\sqrt{a_2 x_2} - \sqrt{b_2 y_2})^2 + \sqrt{a_1 b_1 r_1} + \sqrt{a_2 b_2 r_2}.$$

Thus, $f$ is minimized when $a_1 x_1 = b_1 y_1$ and $a_2 x_2 = b_2 y_2$.

Now suppose that we have an optimal solution to $\mathbf{R_2}$. Clearly the two enclosures must share an entire side of one of them. Assuming that the dimensions of the enclosures are $x_1$ by $y_1$ and $x_2$ by $y_2$ and that the shared side is one in the $y$ direction, we have that the amount of fencing to be minimized is $f = 2x_1 + 2x_2 + y_1 + y_2 + \max\{y_1, y_2\}$, where $x_1 y_1 = r_1$ and $x_2 y_2 = r_2$. If $y_1 < y_2$, the minimum occurs when $y_1 = 2x_1$ and $2y_2 = 2x_2$ or $y_1 = \sqrt{2r_1}$ and $y_2 = \sqrt{r_2}$. But, this minimum is in the given region only if $r_1 < \frac{r_2}{2}$. Under our assumption ($r_1 \geq r_2$), the minimum does not occur when $y_1 < y_2$. However, reversing the subscripts, we see that the minimum does occur at $y_1 = \sqrt{r_1}$ and $y_2 = \sqrt{2r_2}$ when $r_2 < \frac{r_1}{2}$. In the case that $\frac{1}{2} \geq r_2 \geq \frac{r_1}{2}$, the minimum must occur when $y_1 = y_2$. In this case, we have a one parameter problem, see Figure 3. By the "half and half" principle: $3y = 2(\frac{r_1}{y} + \frac{r_2}{y}) = \frac{2}{y}$; giving $y = \sqrt{\frac{2}{3}}$.

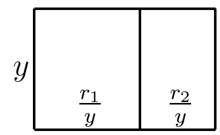



FIGURE 3.

**Research Project 3.** *Investigate the solutions to the problems* $\mathbf{R_n}$.

One might think of problem $\mathbf{R_n}$ as the rectilinear version of the two dimensional soap film problem.

We close this note by observing that all of these research projects have interesting and challenging 3-dimensional analogues.

## Solutions

This section in not intended to be part of the paper; but, simply an aid to the referees in deciding on the viability of this as a research project. We direct our comments to the investigation of the general $\mathbf{C_n}$, $\mathbf{A_n}$ and $\mathbf{R_n}$ problems and we start by developing the concept of a configuration. We define an $n$-arrangement of rectangles to be the union of $n$ rectangles satisfying the condition that their interiors are disjoint.

**Lemma 1.** *All optimal $n$-arrangements for $\mathbf{R_n}$, $\mathbf{A_n}$ and $\mathbf{C_n}$ satisfy the condition that the sides of all rectangles are parallel to one of the axes of a fixed pair of orthogonal axes.*

Proof: Consider an $n$-arrangement. We say that two rectangles are related if their sides are parallel. One easily checks that this is an equivalence relation. Call the union of the rectangles in an equivalence class a *part*. Two different parts have at most a finite number of points in common; hence, one may move two parts independently without increasing the total amount of fencing. Now, if the arrangement has more than one part, we may reorient one part and slide it up to another part decreasing the total amount of fencing used. □

Next we show that the two principles hold for $\mathbf{R_n}$ (and hence for $\mathbf{A_n}$). Let $r_1 \geq r_2 \geq \cdots \geq r_n > 0$ such that $r_1+r_2+\cdots+r_n = 1$. Let $a_0$ be the maximum total area that can be enclosed in $n$ rectangular regions, with areas $r_1a_0, r_2a_0, \cdots, r_na_0$, using a fixed amount of fencing $f_0$ and let $\mathcal{C}_0$ denote the configuration of an optimal solution. Now let $f_1$ denote the minimum amount of fencing needed to enclose $n$ rectangular regions, with areas $r_1a_0, r_2a_0, \cdots, r_na_0$, and let $\mathcal{C}_1$ denote the configuration of an optimal solution. Clearly:

- Since $\mathcal{C}_0$ is a possible configuration for the second problem, $f_1 \leq f_0$.
- Apply the dilation $M = \begin{bmatrix} \frac{f_0}{f_1} & 0 \\ 0 & \frac{f_0}{f_1} \end{bmatrix}$ to $\mathcal{C}_1$ to get $\mathcal{C}_1'$ and note that $\mathcal{C}_1'$ encloses a total area of $a_1 = \frac{f_0^2}{f_1^2}a_0$ in $n$ rectangular regions, with areas $r_1a_1, r_2a_1, \cdots, r_na_1$, using a fixed amount of fencing $f_0$. Thus, $a_1 \leq a_0$ and $f_0 \leq f_1$.

We conclude that $f_1 = f_0$. Of course $\mathcal{C}_1$ need not be identical to $\mathcal{C}_0$, but any configuration optimal for one problem is also optimal for the other.

Suppose that $\mathcal{C}$ is an optimal configuration for one of these problems. Assume that the rectangles are aligned with the axes and let $f =$

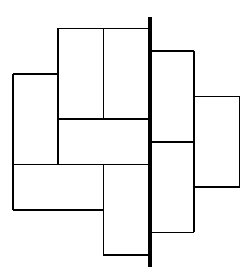
$f_x + f_y$, the fencing in the $x$ and $y$ directions respectively. Apply the transformation $M = \begin{bmatrix} \sqrt{\frac{f_y}{f_x}} & 0 \\ 0 & \sqrt{\frac{f_x}{f_y}} \end{bmatrix}$ to $\mathcal{C}$ to get $\mathcal{C}'$ and note that $\mathcal{C}'$ encloses the same total area in the same ratios with $2\sqrt{f_x f_y}$ fencing. Hence, $2\sqrt{f_x f_y} \geq f_x + f_y$ or $0 \geq (\sqrt{f_x} - \sqrt{f_y})^2$ and we conclude that $f_x = f_y$.

Now we turn our attention to $\mathbf{A_n}$ and the case $n = m^2$.

**Theorem 1.** *The optimal solutions to* $\mathbf{A_n}$ *and* $\mathbf{C_n}$ *require at least* $2(n+\sqrt{n})$ *units of fencing. Furthermore that lower bound will be achieved if only if* $n$ *is a perfect square, in which case both problems have the* $\sqrt{n} \times \sqrt{n}$ *array of unit squares as their unique solution.*

PROOF: The sum of the perimeters of the individual enclosures is greater than or equal to $4n$, with equality if and only if each enclosure is a unit square. The perimeter of the entire configuration encloses an area of $n$ square units. Hence, the perimeter of the entire configuration is greater than or equal to $4\sqrt{n}$ with equality if and only if the entire enclosure is an $\sqrt{n} \times \sqrt{n}$ square. Next, we observe that the sum of the perimeters of the individual enclosures plus the perimeter of the entire configuration is less than or equal to twice the total amount of fencing, with equality if and only if there are no voids (areas entirely surrounded by enclosures but not in any enclosure. Thus the total amount of fencing needed is greater than or equal to $2(n + \sqrt{n})$ with equality if and only if each enclosure is a unit square, the entire enclosure is an $\sqrt{n} \times \sqrt{n}$ square and there are no voids. Clearly these conditions can be met when and only when $\sqrt{n}$ is an integer and the configuration is the $\sqrt{n} \times \sqrt{n}$ array of unit squares.□

Problems $\mathbf{A_5}$ and $\mathbf{C_5}$ are the next unresolved cases. At this point, the investigator will probably wish to prove some lemmas rather than proceed by brute force. A particularly useful result can be formulated to reduce the problem of optimizing an $\mathbf{R_n}$ or $\mathbf{A_n}$ configuration to two smaller problems if the configuration can be divided by a straight fence, as pictured below.

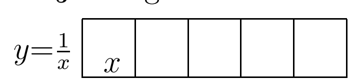


If:

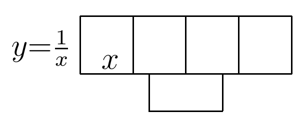
- the configuration on the left side, including the dividing line, has optimal dimensions,
- the configuration on the right side, including the dividing line, has optimal dimensions *as a configuration against a barn*;
- and if the length of portion of the dividing line used by the right side is less than or equal to the length of portion of the dividing line used by the left side,

then the total configuration has optimal dimensions. If this last inequality does not hold, then one may try reversing the roles of the left and right. If the inequality still does not hold then one may add the constraint that the two sides match perfectly along that line.

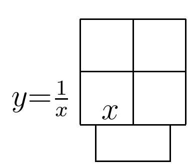
This result can be used to show that if the rectangles of an optimal configuration for $\mathbf{A_n}$ are in a straight line then the rectangles are all congruent. So the first $\mathbf{A_5}$ configuration we consider is:



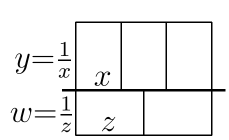

The dimensions are obtained by solving $\frac{6}{x} = 10x$ to get $x = \sqrt{\frac{3}{5}}$ and $y = \sqrt{\frac{5}{3}}$ with the total length of fencing equal to $4\sqrt{15} \approx 15.4919$. The next configuration to consider is:



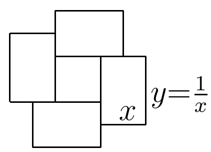

Here, using the "divide and conquer" result, we see that the dimensions of the four top rectangles are $x = \sqrt{\frac{5}{8}}$ and $y = \sqrt{\frac{8}{5}}$ with the total length of fencing equal to $4\sqrt{10}$. The bottom rectangle is $\sqrt{2} \times \frac{\sqrt{2}}{2}$ and contributes $2\sqrt{2}$ for a total of $4\sqrt{10} + 2\sqrt{2} \approx 15.4775$ units of fencing. The next configuration will be even better:




Here the four top rectangles are unit squares while the bottom rectangle contributes $2\sqrt{2}$ for a total of $12 + 2\sqrt{2} \approx 14.8284$ units of fencing. The results for the next configuration are somewhat of a surprise:



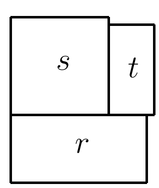
In this case, the lengths of both subconfigurations along the dividing line are equal and the rectangles are all congruent: For the top (as a stand alone configuration), we have $\frac{4}{x} = 6x$ giving $x = \sqrt{\frac{2}{3}}$ and $y = \sqrt{\frac{3}{2}}$; for the bottom (as an "against the barn" problem), we have $\frac{3}{z} = 2z$ giving $w = x = \sqrt{\frac{2}{3}}$ and $z = y = \sqrt{\frac{3}{2}}$! The total amount of fencing used is $6\sqrt{6} \approx 14.6969$ - the best yet. This last configuration is in fact the optimal configuration and therefore the solution to $\mathbf{C_5}$ as well! [The authors expected that, like $\mathbf{A_3}$ and $\mathbf{C_3}$, $\mathbf{A_5}$ and $\mathbf{C_5}$ would not have a common solution.] But, before we can conclude that the last configuration is optimal, we must consider the possibility of a configuration that cannot be divided into two parts. It is easy to convince oneself that, with congruent rectangles, such a configuration is impossible. Hence the above configuration is the solution to $\mathbf{C_5}$. However, for $\mathbf{A_5}$, such a configuration is possible. Using symmetry arguments the configuration may be seen to have the following form:




where the center enclosure is the unit square. The amount of fencing is $\frac{4}{x} + 8x + 4$ and that is minimized when $x = \frac{\sqrt{2}}{2}$ and $y = \sqrt{2}$ for a total of $8\sqrt{2} + 4 \approx 15.3137$.

We conclude this discussion of solutions by outlining the solution to $\mathbf{R_3}$. We will assume that we are to construct three rectangular enclosures with areas $r$, $s$ and $t$, where $r \geq s \geq t$ and $r+s+t = 1$, while minimizing the amount of fence used. With rather simple arguments, we can show that one can always find an optimal solution with a cluster configuration (as opposed to a three in a row configuration). The six possibilities with the rectangle of area $r$ on the bottom are listed below. A configuration cannot be optimal if any of the inequalities listed with it are violated.

**Configuration $\mathcal{C}_1$:**

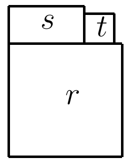



We have: $\sqrt{s} \geq \sqrt{2}t$, which gives $r \geq 1 - \frac{3}{2}s$.

And: $\sqrt{s} + \sqrt{\frac{t}{2}} \geq \sqrt{2r}$, which gives $r \leq \frac{1+s}{5}$ or $25r^2 - 10r - 2rs + 9s^2 - 6s + 1 \leq 0$.

The fencing used is $f = 4\sqrt{s} + \sqrt{8r} + \sqrt{8t}$.

**Configuration $\mathcal{C}_2$:**

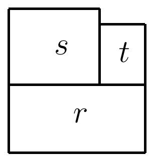



We have: $\sqrt{s} \geq \sqrt{2t}$, which gives $r \geq 1 - \frac{3}{2}s$.

And: $\sqrt{r} \geq \sqrt{2s} + \sqrt{t}$, which gives $r \geq \frac{1+s}{2}$ and $4r^2 + 9s^2 + 4rs - 4r - 6s + 1 \geq 0$.

The fencing used is $f = 4\sqrt{r} + \sqrt{8s} + 2\sqrt{t}$.

**Configuration $\mathcal{C}_3$:**



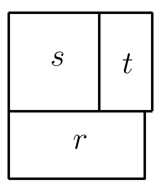

We have: $\sqrt{s} \geq \sqrt{2t}$, which gives $r \geq 1 - \frac{3}{2}s$.

And: $\sqrt{s} + \sqrt{\frac{t}{2}} \leq \sqrt{2r}$, which gives $r \leq \frac{1+s}{5}$ and $25r^2 - 10r - 2rs + 9s^2 - 6s + 1 \geq 0$.

And: $\sqrt{r} \leq \sqrt{2s} + \sqrt{t}$, which gives $r \leq \frac{1+s}{2}$ or $4r^2 + 9s^2 + 4rs - 4r - 6s + 1 \leq 0$.

The fencing used is $f = \sqrt{24 - 12t + 24\sqrt{2st}}$.

**Configuration $\mathcal{C}_4$:**

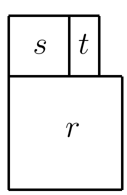



We have: $\sqrt{s} \leq \sqrt{2t}$, which gives $r \leq 1 - \frac{3}{2}s$.

And: $\sqrt{\frac{3}{2}(1-r)} \geq \sqrt{2r}$, which gives $r \leq \frac{3}{7}$.

The fencing used is $f = \sqrt{24(1-r)} + \sqrt{8r}$.

**Configuration $\mathcal{C}_5$:**

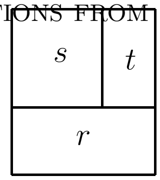



We have: $\sqrt{s} \leq \sqrt{2t}$, which gives $r \leq 1 - \frac{3}{2}s$.

And: $\sqrt{r} \leq \sqrt{3(1-r)}$, which gives $r \geq \frac{3}{4}$.

The fencing used is $f = 4\sqrt{r} + 2\sqrt{3 - 3r}$.

**Configuration $\mathcal{C}_6$:**




We have: $\sqrt{s} \leq \sqrt{2}t$, which gives $r \leq 1 - \frac{3}{2}s$.
And: $\sqrt{r} \leq \sqrt{\frac{3}{2}(1-r)} \leq \sqrt{2r}$, which gives $r \geq \frac{3}{7}$.
And: $\sqrt{r} \geq \sqrt{3(1-r)}$, which gives $\frac{3}{4} \geq r$.
The fencing used is $\quad f = 2\sqrt{9-3r}$.

It is clear that different configurations will be optimal among these six for different values of the parameters. We will consider $r$ and $s$ as independent variables and picture the ranges in which these configurations are viable in the $rs$-plane (Figures 1, 2 and 3 below). We note that the inequalities $r \geq s$, $s \geq t$ and $t \geq 0$ define the triangular region bounded by the lines $r = s$, $r = 1 - 2s$ and $r = 1 - s$. This region is outlined in heavy lines in each of the Figures.

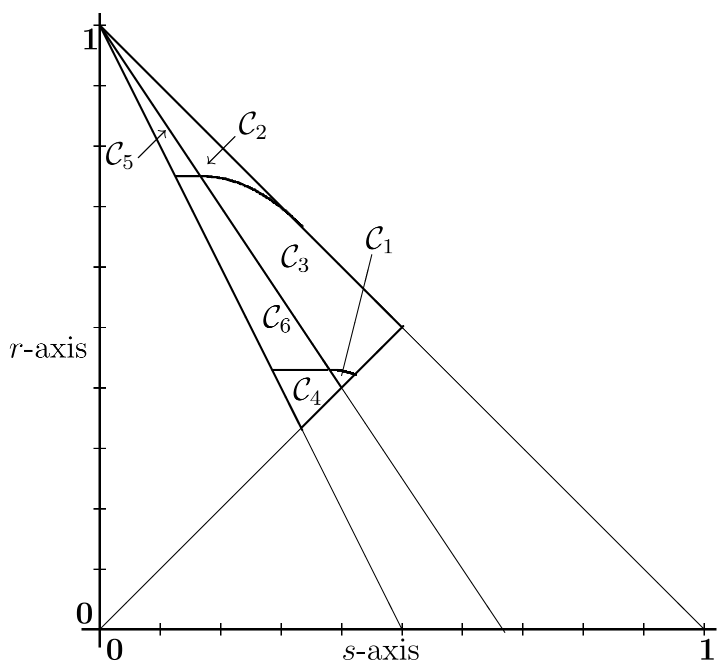



FIGURE 4.

In Figure 4, this basic region is further partitioned into six subregions and each of these regions is labeled by the configuration that is valid in that region: the basic region is divided into regions by the line $\sqrt{s} = \sqrt{2t}$ or $r = 1 - \frac{3}{2}s$; the subregion to the left of this line is further divided by the horizontal lines $r = \frac{3}{4}$ and $r = \frac{3}{7}$, $\mathcal{C}_4$ being valid in the bottom subregion, $\mathcal{C}_6$ being valid in the central subregion and $\mathcal{C}_5$ being valid in the top subregion; the subregion to the right of the line $r = 1 - \frac{3}{2}s$ is further divided by arcs of the ellipses $25r^2 - 10r - 2rs + 9s^2 - 6s + 1 = 0$ and $4r^2 + 9s^2 + 4rs - 4r - 6s + 1 = 0$, $\mathcal{C}_1$ being valid in the bottom subregion, $\mathcal{C}_3$ being valid in the central subregion and $\mathcal{C}_2$ being valid in the top subregion.

We must also consider each of these configurations with all permutations of $r$, $s$ and $t$ which are consistent with the restricting inequalities. This yields four further configurations labeled by $\mathcal{C}'_1$, $\mathcal{C}'_3$, $\mathcal{C}'_4$ and $\mathcal{C}''_4$. These configurations and the appropriate inequalities and fencing formula are listed below. Configurations $\mathcal{C}'_1$, $\mathcal{C}'_3$ and $\mathcal{C}'_4$ partition our basic region. In Figure 2, we label each subregion by the configuration, in this list, which valid in that region. Configuration $\mathcal{C}''_4$ is valid below the line $r = 2s$ and is pictured only in the last figure.

**Configuration $\mathcal{C}'_1$:**
We have: $\sqrt{r} \geq \sqrt{2}t$, which gives $r \geq \frac{2}{3}(1-s)$.
And: $\sqrt{r} + \sqrt{\frac{t}{2}} \geq \sqrt{2s}$, which gives $r \geq 5s - 1$ or $0 \geq 25s^2 - 10s - 2rs + 9r^2 - 6r + 1$. The fencing used is $f = 4\sqrt{r} + \sqrt{8s} + \sqrt{8t}$.

**Configuration $\mathcal{C}'_3$:**
We have: $\sqrt{r} \geq \sqrt{2}t$, which gives $r \geq \frac{2}{3}(1-s)$.
And: $\sqrt{r} + \sqrt{\frac{t}{2}} \leq \sqrt{2s}$, which gives $r \leq 5s - 1$ and $0 \leq 25s^2 - 10s - 2rs + 9r^2 - 6r + 1$.
And: $\sqrt{s} \leq \sqrt{2r} + \sqrt{t}$, which holds through out the basic region (precluding a $\mathcal{C}'_2$ subregion).
The fencing used is $f = \sqrt{24 - 12t + 24\sqrt{2rt}}$.

**Configuration $\mathcal{C}'_4$:**
We have: $\sqrt{r} \leq \sqrt{2}t$, which gives $r \leq \frac{2}{3}(1-s)$.
And: $\sqrt{\frac{3}{2}(1-s)} \geq \sqrt{2s}$, which gives $s \leq \frac{3}{7}$.
The fencing used is $f = \sqrt{24(r+t)} + \sqrt{8s}$.

**Configuration $\mathcal{C}''_4$:**
We have: $\sqrt{r} \leq \sqrt{2}s$, which gives $r \leq 2s$. And: $\sqrt{\frac{3}{2}(1-t)} \geq \sqrt{2t}$ which gives $t \leq \frac{3}{7}$ (always satisfied in our basic region).
The fencing used is $f = \sqrt{24(r+s)} + \sqrt{8t}$.

In Figure 3 below, we have superimposed the decompositions from the two sets of configurations resulting in a partition of the basic region

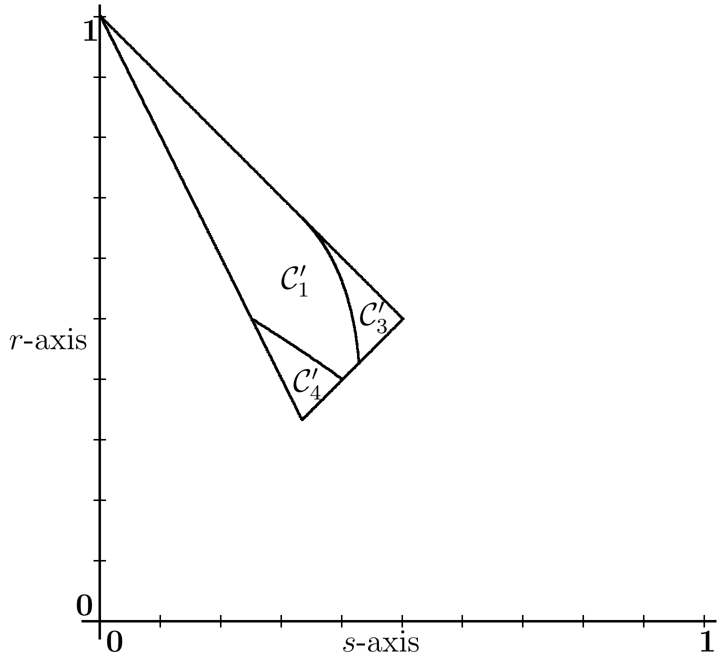



FIGURE 5.

into eleven cells. We have labeled these cells by $a$ through $k$. In each of these cells there are two or three candidates for the optimal configuration, one $\mathcal{C}_i$, one $\mathcal{C}_i'$ and, in the regions or parts of regions below the line $r = 2s$, $\mathcal{C}_4''$. The possible configurations and the optimal configuration for each cell are recorded in the table following Figure 3.

| *Region(s)* | *Possible Configurations* | *Optimal Configuration* |
|---|---|---|
| $a$ | $\mathcal{C}_5,\ \mathcal{C}_1'$ | $\mathcal{C}_5$ |
| $b$ | $\mathcal{C}_2,\ \mathcal{C}_1'$ | $\mathcal{C}_2$ |
| $c$ | $\mathcal{C}_6,\ \mathcal{C}_1'$ | $\mathcal{C}_6$ |
| $d$ | $\mathcal{C}_3,\ \mathcal{C}_1'$ | $\mathcal{C}_3$ |
| $e$ | $\mathcal{C}_6,\ \mathcal{C}_1',\ \mathcal{C}_4''$ | $\mathcal{C}_6$ |
| $f$ | $\mathcal{C}_3,\ \mathcal{C}_1',\ \mathcal{C}_4''$ | $\mathcal{C}_3$ |
| $g$ | $\mathcal{C}_3,\ \mathcal{C}_3',\ \mathcal{C}_4''$ | $\mathcal{C}_3$ |
| $h$ | $\mathcal{C}_6,\ \mathcal{C}_4',\ \mathcal{C}_4''$ | $\mathcal{C}_6$ |
| $i$ | $\mathcal{C}_4,\ \mathcal{C}_4',\ \mathcal{C}_4''$ | $\mathcal{C}_4$ |
| $j$ | $\mathcal{C}_4,\ \mathcal{C}_1',\ \mathcal{C}_4''$ | $\mathcal{C}_4$ |
| $k$ | $\mathcal{C}_1,\ \mathcal{C}_1',\ \mathcal{C}_4''$ | $\mathcal{C}_1$ |

Since the configuration with the $r$ region on the bottom is always the optimal, the solution space is best described by Figure 1. Perhaps

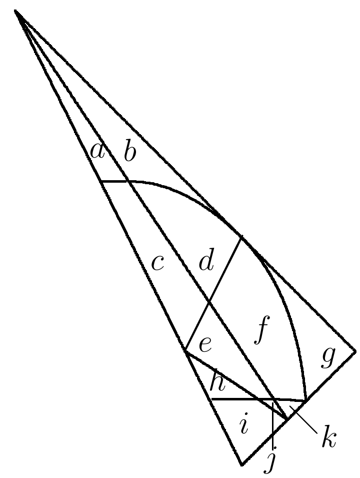



FIGURE 6.

a clever student researcher will be able to eliminate the $\mathcal{C}'$ and $\mathcal{C}''$ configurations at the outset and greatly simplify this solution.

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