This talk is a report of joint work done with Mark Watkins [2]. It represents the start of an extensive project to study highly symmetric planar infinite graphs. Edge-transitivity is one of the strongest symmetry conditions that one can impose on a graph and it is our point of departure. The precise set of conditions I will impose for this talk is: locally finiteness, planarity, 3-connectivity, edge-transitivity and that all faces be finite.

By Whitney's Theorem [9] (or the infinite form of Whitney's Theorem [6]), 3-connected implies "unique embedding" and "unique dual". We believe that the condition that all faces be finite follows from the other conditions. But, in order to simplify the discussion, I will assume here that all faces are finite.

One useful concept in the study of infinite graphs is that of the *ends* of a graph. These ends were defined by Halin [4] for an arbitrary graph. If the graph is planar one may visualize them as the limit points of the graph when embedded on sphere. This concept gives a natural tripartite classification of graphs:

> 0-ended (finite),
> 1-ended (planar embedding without limit points) and
> multi-ended

The graphs under consideration here will have 0, 1, 2 or uncountably many ends. The exclusion of the values between 2 and "uncountably many obtained by combining results of Halin [5] and Jung [7].

Prior to our investigation, the following information was known:

> There are exactly 9, 0-ended, planar, 3-connected, edge-transitive, graphs: the five platonic graphs, the 1-skeleton of the cuboctahedron and its dual, the 1-skeleton of the icosidodecahedron and its dual.
>
> The 1-ended planar, 3-connected, edge-transitive, graphs were characterized by Grunbaum and Shephard [3].
>
> The 2-ended, planar, 3-connected, edge-transitive, graphs were characterized by Watkins [8].

Before we can state the Grunbaum and Shephard result, we must introduce the *edge symbol*: <v,w;c,d>, ($v \leq w$ and $c \leq d$), of an edge e: v and w are the valences of the endvertices of e while c and d are the valences of the two faces incident with e. Since the embedding is unique, the edge symbol of e is well defined. We say G is *edge- homogenous* if all of its edges have same edge symbol.

If G is a 3-connected, edge-homogenous graph with edge symbol <v,w;c,d>, then the following three conditions hold:

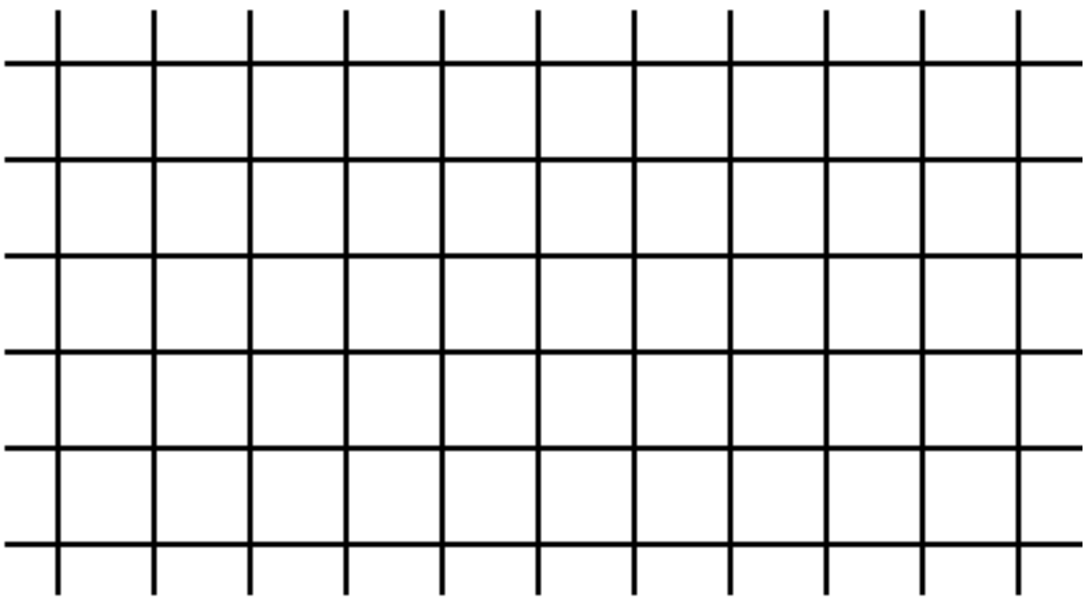
$v,w,c,d \geq 3$;
v or w odd implies $c=d$;
c or d odd implies $v=w$.

The Grunbaum and Shephard's result: If v,w,c,d satisfy the above conditions, a 3-connected, edge-homogenous planar graph with edge symbol <v,w;c,d> exists. Furthermore:

If $1/v + 1/w + 1/c + 1/d > 1$, then G is edge-transitive and finite (and hence one of the 9 finite graphs listed above).

If $1/v + 1/w + 1/c + 1/d = 1$ *and G is 1-ended*, then G is edge-transitive and, in fact, one of the 5 edge-transitive tesselations of the plane. See, for example Figures 1 and 2.

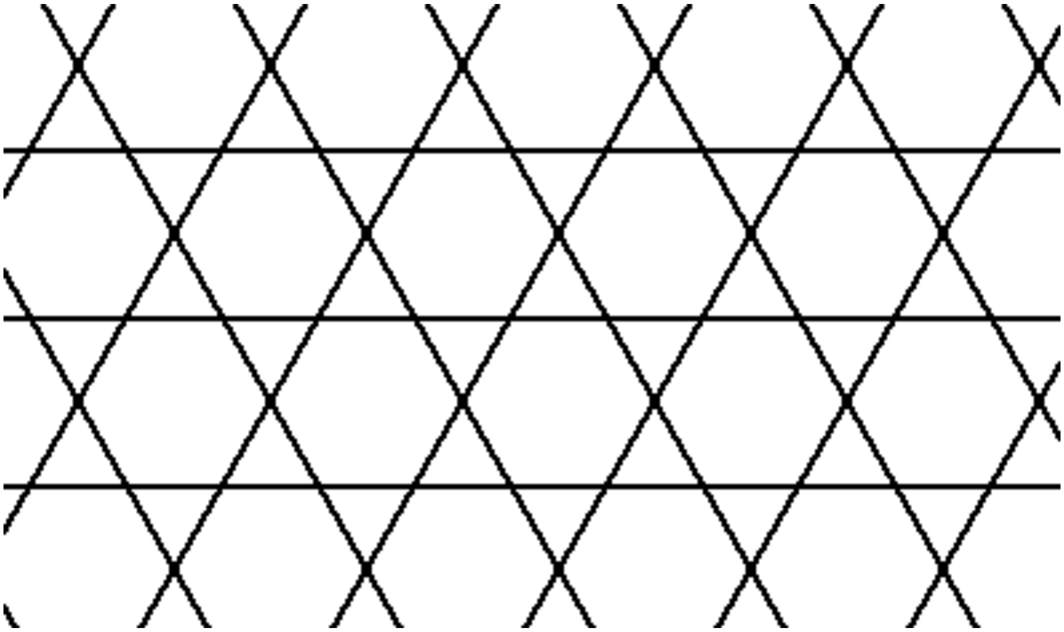
If $1/v + 1/w + 1/c + 1/d < 1$ *and G is 1-ended*, then G is edge-transitive and unique (with respect to its edge symbol). See, for example, Figure 3.



Edge symbol <4,4;4,4>

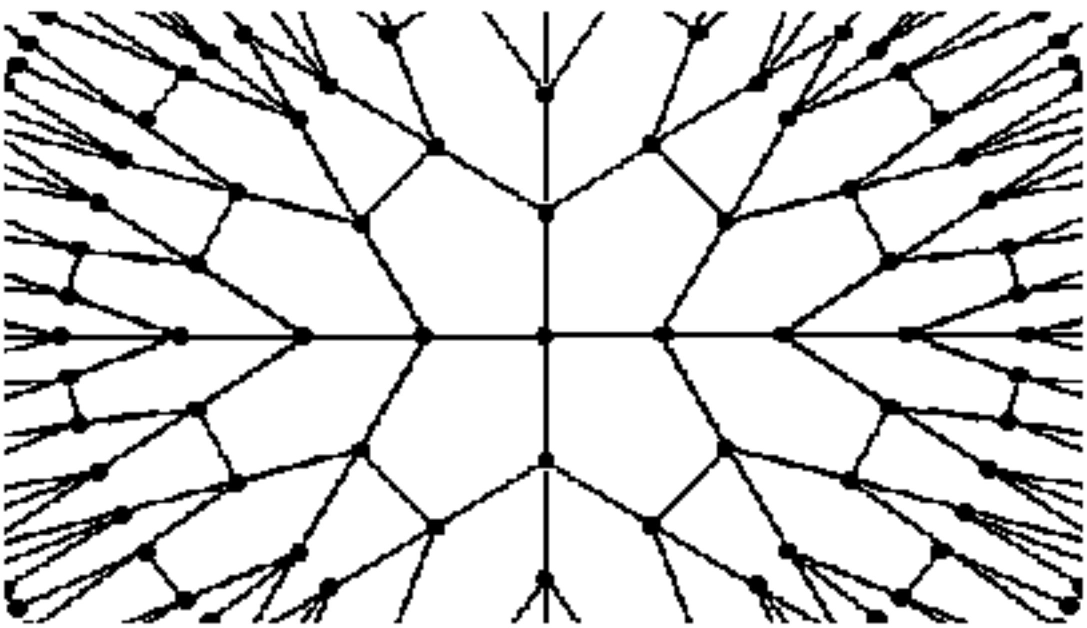
Figure 1

The 2-ended, planar, 3-connected, edge-transitive, graphs (as characterized by Watkins), all have edge symbol <4,4;4,4> and are constructed by identifying the sides of a strip of the tesselation with this edge symbol (Figure 1) bounded by two parallel 45 degree lines. See Figure 4. Furthermore, one easily checks that the planar tesselation and the graphs obtained by this construction are all of the planar, 3-connected, edge-transitive, graphs with edge symbol <4,4;4,4>.



Edge symbol <4,4;3,6>

Figure 2



Edge symbol <4,4;5,5>

Figure 3

Note that, in the 2-ended case, the edge symbol no longer characterizes the graph. However, the edge symbol plus the *diameter* of the cylinder does characterize the graph.

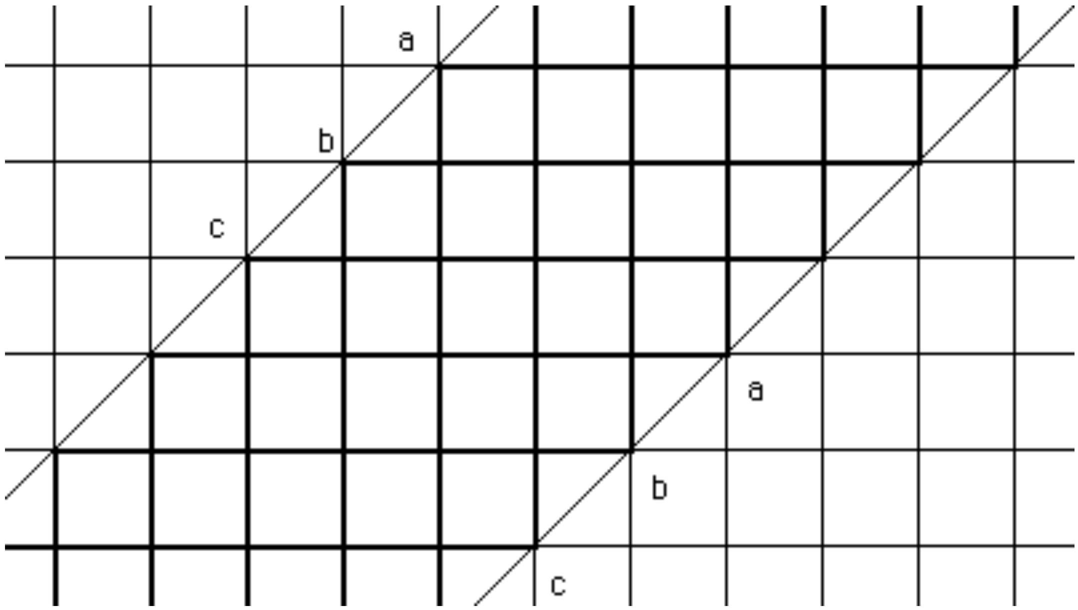



Figure 4

One of the first questions we considered was: Do infinite-ended, planar, 3-connected, edge-transitive, graphs exist? The answer is yes. In fact they exist in profusion due to a very general construction called *interleaving*.

Let G and H be locally finite, planar, edge-transitive, graphs with edge symbols <v,v;c,h> and <w,w;d,h>, respectively, where both v and w are even. We may then combine these graphs to produce a locally finite, planar, edge-transitive, multi-ended graph with edge symbol <v,w,2c,2d>. This graph is denoted by G|H. We illustrate the construction of G|H with G a 3-cycle whose edges have been doubled (edge symbol <4,4;2,3>) and H a copy of the octahedral graph (edge symbol <4,4;3,3>).

The first step in the interleaving construction is to partition the faces of H and G into two classes so that around each vertex the faces alternate classes (since all vertices are even the dual graph is bipartite). In each graph, color the faces in the class with covalence c or d black and the faces in the class with covalence h white. Then insert a two valent vertex in the center of each edge. See Figure 5.

Now "replace" the white faces (including the outside face) of a copy of H with copies of G. This is done in Figure 6. Next replace the white faces of this graph with copies of H and so on, interleaving copies of G and H. It is not too difficult to prove that G|H is edge transitive; it is clearly locally finite, planar and 3-connected. Also one easily sees that G|H has more than two ends and hence must have infinitely many ends.

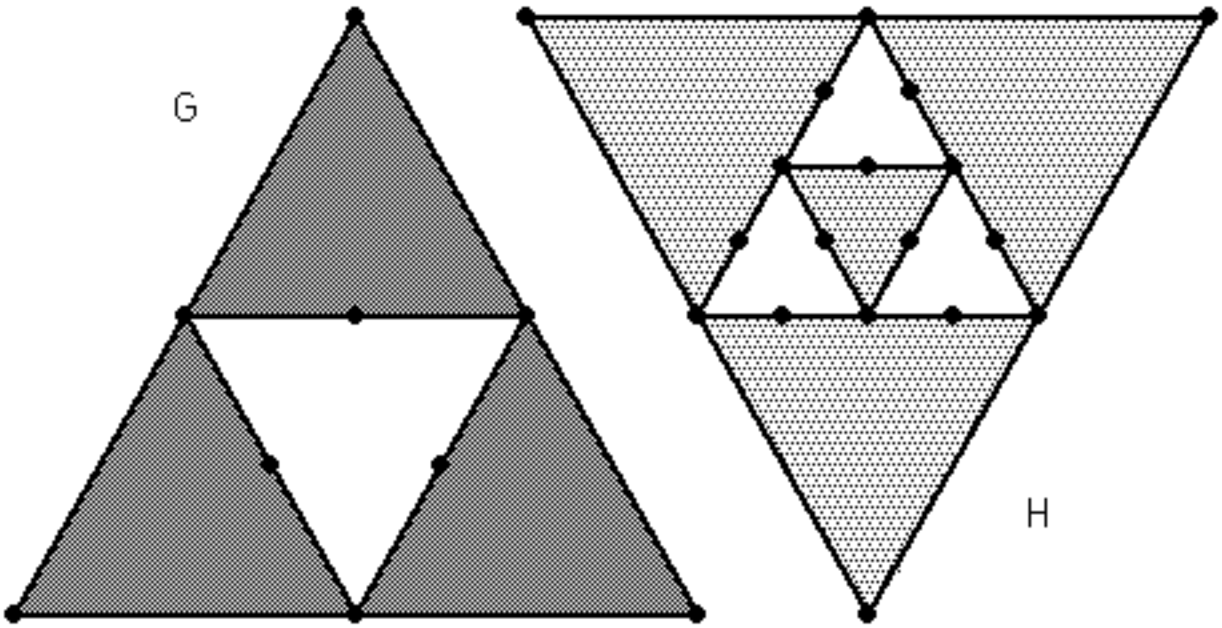



Figure 5

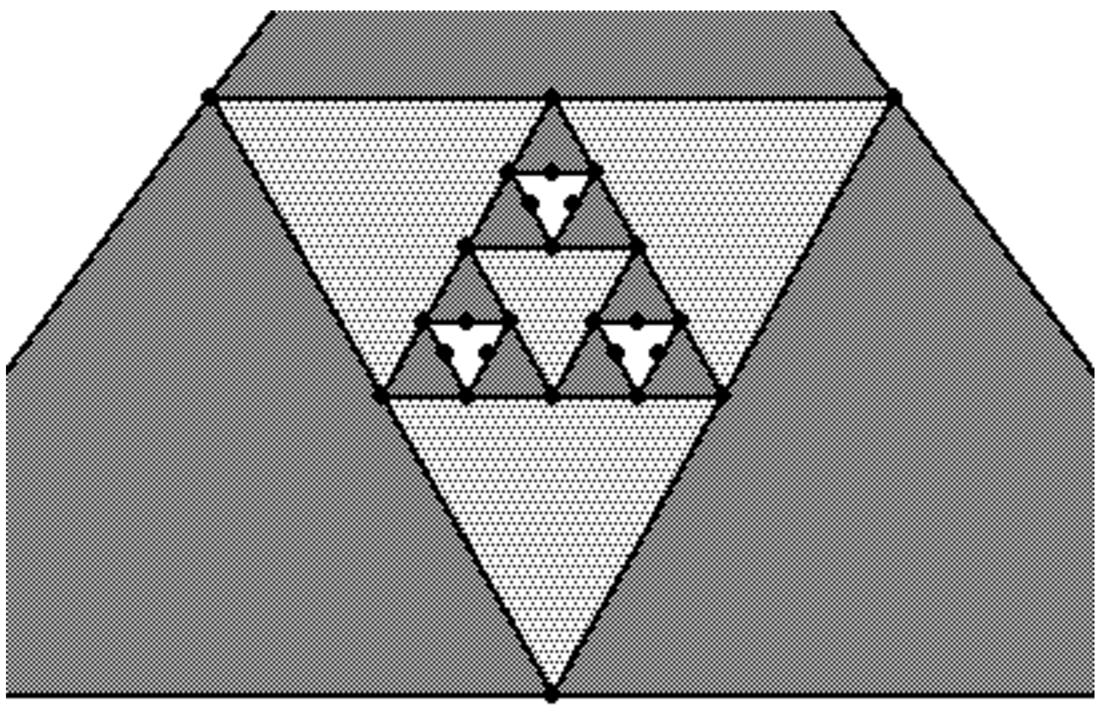


Figure 6

The next natural question to ask is: Can the infinite-ended, planar, 3-connected, edge-transitive, graphs be characterized? In particular, are they all obtained by the interleaving construction? We be believe the answer to be yes.

The fundamental tools that we use in our investigation of an infinite-ended planar, 3-connected, edge-transitive, graph G are:

the structure of the automorphism group of G and
a new set of objects in G called *Petrie walks*.

To construct a Petrie walk in an arbitrary planar graph: move along an edge then turn sharp right, then sharp left, then sharp right, etc.; return to initial edge and proceed in opposite direction in the same way. See Figure 7, below. Also, one may interchange left and right to get a second (not obviously distinct) Petrie walk through the same initial edge. Petrie walks in finite graphs were studied by Coxeter [1]. The reader may wish to construct a few Petrie Walks in the graphs in Figures 1 through 4.

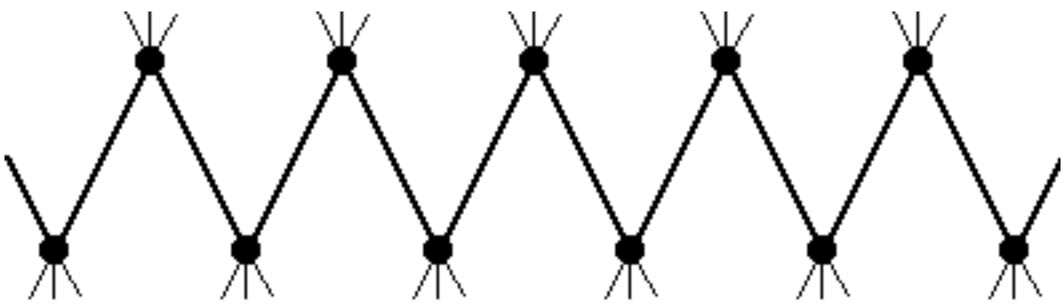


Figure 7

Properties of Petrie walks:

- Petrie walks are "self dual" [easy to see - Figure 8].
- Each edge belongs to exactly two Petrie walks ["at most two" is easy to show - Figure 9, "exactly two" is hard to show].
- There are at most two Petrie transitivity classes [follows from edge-transitivity].
- A Petrie walk is either an elementary circuit (of even length) or a ray - elementary 2-way infinite path [hard to show].

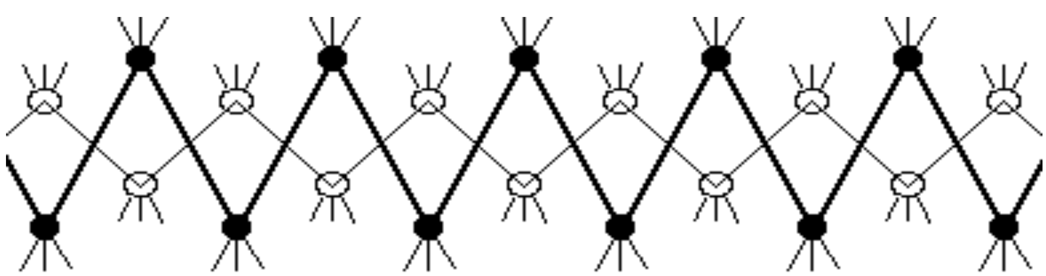


Figure 8

Figure 9

There are at most two Petrie transitivity classes [follows from edge-transitivity]. Edge transitivity and the fact that each Petrie walk is either an elementary circuit or a ray leads to the *extended edge symbol*, <v,w;c,d;h,k>, (v≤w, c≤d and h≤k≤∞), where h and k are the lengths of the Petrie walks through the edge; and, to a new tripartite classification:

*circuit type* <v,w;c,d;h,k> (h≤k<∞);
*ray type* <v,w;c,d;∞,∞>;
*mixed type* <v,w;c,d;h,∞>, (h<∞).

The graphs in Figures 1 and 3 are of ray type with extended edge symbols of <4,4;4,4;∞,∞> and <4,4;5,5;∞,∞> respectively; the graph in Figure 2 is of circuit type with extended edge symbol <4,4;3,6;12,12> ; and the graph in Figure 4 is of mixed type with extended edge symbol <4,4;4,4;6,∞>. Graphs resulting from the interleaving construction are of mixed type. The h-faces about which the interleaving is carried out become Petrie cycles of length 2h. It will then follow from our characterization of graphs of circuit type (below) that these graphs are indeed of mixed type.

Does the tripartite classifications based on Petrie walks coincide with the tripartite classification based on the number of ends? The answer is: "Almost." Clearly, all 9 of the 0 ended (finite) graphs are of circuit type; however, the graph in Figure 2 is infinite but also of circuit type. We conjecture that, except for a very special class of graphs containing 2, "0-ended" corresponds to "circuit type", "1-ended" corresponds to "ray type" and "multi-ended" corresponds to "mixed type".

The following results go a long way toward verifying this correspondence.

**Theorem** If two Petrie walks in G cross more than once, then G is Petrie transitive.

**Corollaries**

Graphs of circuit type are Petrie transitive.
Graphs of mixed type are multi-ended.

**Theorem** Infinite graphs of circuit type are one-ended with extended edge symbol <4,4,3,h,2h,2h> or <3,h,4,4,2h,2h> ($h>5$). In either case, the graph is uniquely determined by its extended edge symbol.

As noted above, the graph in Figure 2 has extended edge symbol <4,4;3,6;12,12> and is a representative of the class of one-ended graphs described in this theorem.

Combining Watkins' result [8] with the above results gives us the following relation between the two tripartite classifications:

| | 0-ended | 1-ended | multi-ended |
|---|---|---|---|
| circuit type | yes | yes* | no |
| ray type | no | yes | no |
| mixed type | no | no | yes |

*This is the exceptional class of graphs described in the above theorem.

Returning to the question, "Are all mixed type graphs constructed by interleaving?", we introduce the concept of *deconstruction.*

**Deconstruction**: Let K be one of our graphs of mixed type with extended edge symbol <2v,2w;2c,2d;2h,∞> (It is not difficult to show that, if G is of mixed type, then all of the parameters of its edge symbol must be even.)

1) Fix a Petrie cycle $C_{0.}$
2) Designate one side of $C_0$ to be the inside of $C_{0.}$
3) Let Δ = {Petrie cycles outside of $C_0$ not separated from $C_0$ by another Petrie cycle}.
4) For each C in Δ, choose its inside so that $C_0$ is on its outside.
5) Delete the inside of $C_0$ and of each C in Δ.
6) Smooth over all vertices of valence 2.

The resulting graph G will be either a circuit of double edges or a locally finite planar, 3-connected, edge-transitive, (multi)graph; its edge symbol will be one of the edge symbols from one of the following pairs:

<2v,2v;c,h> and <2w,2w;d,h>,
<2v,2v;d,h> and <2w,2w;c,h>.

Now reverse the inside and outside of $C_0$ and repeat steps 2) through 6). The resulting graph H will be either a circuit of double edges or a locally finite planar, 3-connected, edge-transitive, (multi)graph; its edge symbol will be the other edge symbols from the same pair. Then K = G|H. We summarize this with the following tree diagram:

K:<2v,2w;2c,2d;2h,∞>
/ \
G:<2v,2v;c,h;?,?> H:<2w,2w;d,h;?,?>

If either G or H is of mixed type, it would deconstruct also. Hence, to each graph of mixed type, we associate a deconstruction tree with pendant nodes representing graphs of circuit or ray type, all other nodes representing graphs of mixed type. When this deconstruction tree is finite, it characterizes the graph. But, is this tree always finite? The answer seems to be no.

Suppose K has extended edge symbol <2v,2v;2c,2d;2h,∞> where d=2c and h=2d. Then the following seems possible:

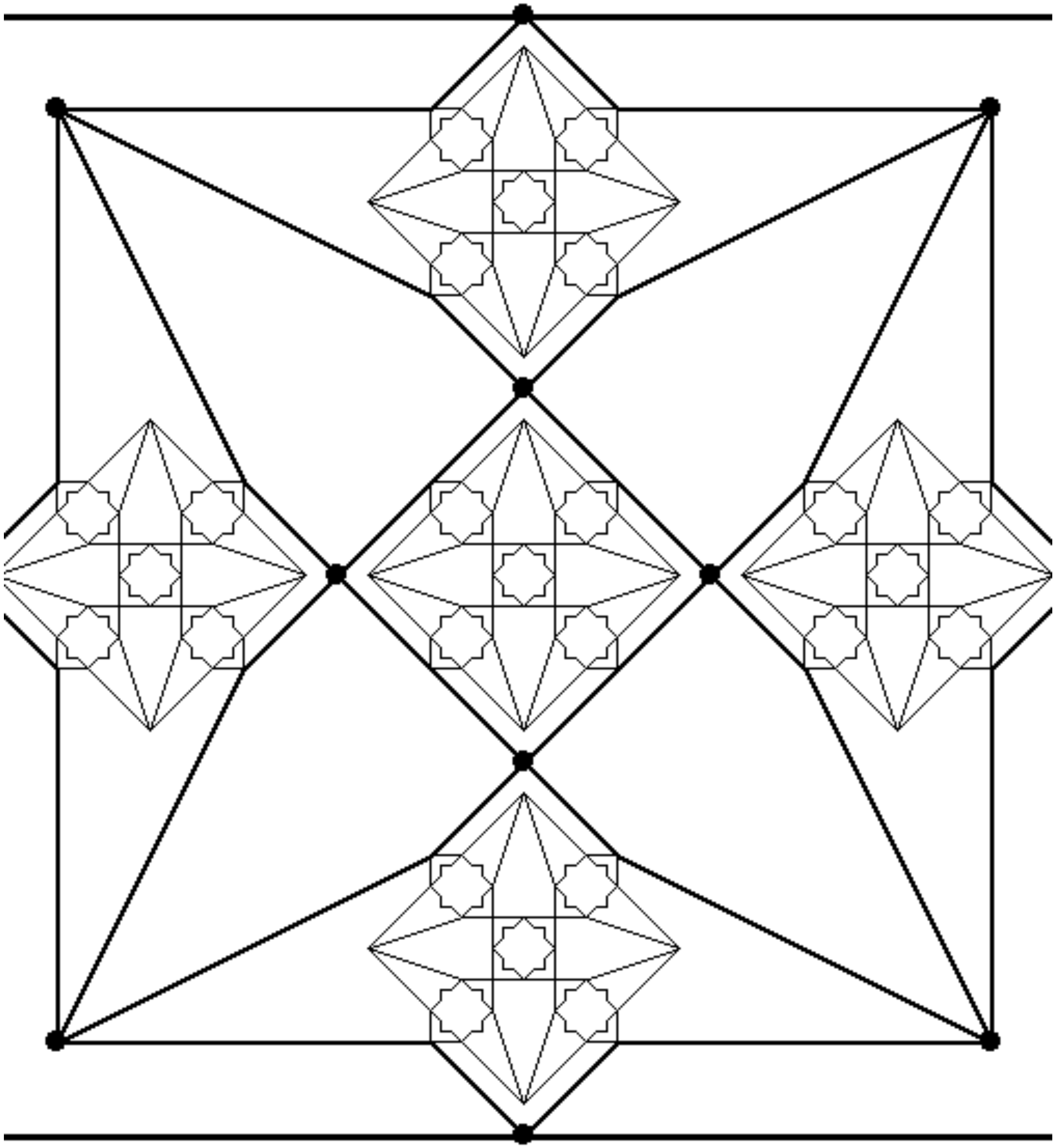
K:<2v,2v;2c,2d;2h,∞>
/ \
G:<2v,2v;c,h;?,?> H:<2v,2v;d=2c,h=2d;2h,∞>

Thus, we could have H=K and a graph satisfying K = G|K might exist. The deconstruction tree of such a graph would be infinite.

In Figure 10, we include an additional infinite ended graph of mixed type. To "complete" the portion of the graph drawn here, the innermost 16-gons (which are slightly distorted here) should be filled with copies of the outermost 16-gon.

The interested reader may wish to deconstruct this graph and produce its deconstruction tree.

This example illustrate another feature of these graphs: many of them lead to *self-similar* tilings of the plane. The tiling represented here (once the distortion of the innermost 16-gons are corrected) uses tiles from three similarity classes of hexagonal tiles (one tile is a pentagon with a vertex in the center of one edge).



Extended edge symbol <4,4;6,6;8.∞>

Figure 10

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