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Graph Theory and Reliability

You May Rely on the Reliability Polynomial for Much More Than You Might Think

JACK GRAVER AND MILTON SOBEL

Department of Mathematics, Syracuse University, Syracuse, New York, USA

The reliability polynomial $R_{\mathscr{S}}(p)$ of a collection $\mathscr{S}$ of subsets of a finite set X has been extensively studied in the context of network theory. There, X is the edge set of a graph (V, X) and $\mathscr{S}$ the collection of the edge sets of certain subgraphs. For example, we may take $\mathscr{S}$ to be the collection of edge sets of spanning trees. In that case, $R_{\mathscr{S}}(p)$ is the probability that, when each edge is included with the probability p, the resulting subgraph is connected. The second author defined $R_{\mathscr{S}}(p)$ in an entirely different way enabling one to glean additional information about the collection $\mathscr{S}$ from $R_{\mathscr{S}}(p)$. Illustrating the extended information available in the reliability polynomial is the main focus of this article while demonstrating the equivalence of these two definitions is the main theoretical result.

Keywords Inclusion–exclusion; Reliability polynomial.

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1. Introduction

Let X be an n-element set and let $\mathscr{S}$ be a collection of subsets of X. If no subset in $\mathscr{S}$ contains any other subset in $\mathscr{S}$, we call $\mathscr{S}$ a clutter. The collection $\mathscr{S}^b$ is defined to be the collection of the minimal subsets (under set inclusion) in the collection of all subsets that have a non empty intersection with every subset in $\mathscr{S}$. The subsets in $\mathscr{S}^b$ are called the blocking sets for $\mathscr{S}$ and the operator $(\ )^b$ is a duality operator in that, for clutters, $\mathscr{S}^{bb} = \mathscr{S}$. This duality is the basis of the famous Max-Flow-Min-Cut Theorem: if $\mathscr{S}$ are the edge sets of the flows through a network, $\mathscr{S}^b$ is the collection of cuts (see Edmonds and Fulkerson, 1970).

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Address correspondence to Jack Graver, Department of Mathematics, Syracuse University, Syracuse, NY 13244-1150, USA; E-mail: jegraver@syr.edu

Associated with each collection $\mathcal{S}$ is its reliability polynomial $R_{\mathcal{S}}(p)$ defined by:

$$R_{\mathcal{S}}(p) = \sum_{i=0}^{|X|} n_i p^i (1-p)^{|X|-i},$$

where n_i is the number of i-element subsets of X that contain some set in $\mathcal{S}$. One interprets the value of $R_{\mathcal{S}}$ at $p \in [0, 1]$ as follows: Let $Y \subseteq X$ be the result of including each element of X with probability p; then $R_{\mathcal{S}}(p)$ is the probability that Y will contain a set from $\mathcal{S}$. The reliability polynomial has been extensively studied in the context of network theory. There X is the edge set of a graph (V, X) and $\mathcal{S}$ the collection of the edge sets of certain subgraphs, for example, the spanning trees. In the case of spanning trees, $R_{\mathcal{S}}(p)$ is the probability that the subgraph (V, Y) will be connected. See Colbourn (1987, 1993) for an introduction to network reliability and its literature.

An interesting and useful fact is that the reliability polynomials of $\mathcal{S}$ and $\mathcal{S}^b$ are related by:

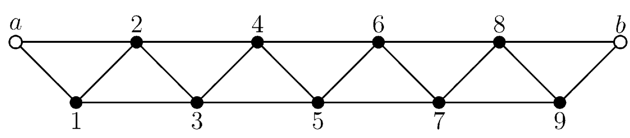
$$R_{\mathcal{S}^b}(p) = 1 - R_{\mathcal{S}}(1-p).$$

The coefficients of these polynomials clearly encode some of the structure of the collection and its blocking collection. Demonstrating that the information encoded in these coefficients may be capitalized upon in a variety of other probability and combinatorial settings is the main purpose of this article.

Throughout the article, we will illustrate our definitions and results with specific cases from the following class of examples. Consider the graph (V, E) where $V = \{a, 1, 2, \ldots, n, b\}$ and where the edge set E is defined as follows:

$$E = \{(a, 1), (a, 2), (1, 2), (1, 3), \ldots, (i, i+1), (i, i+2), \ldots, (n-1, n), (n-1, b), (n, b)\}.$$

The case $n = 9$ is illustrated below. We take $X = \{1, 2, \ldots, n\}$ and $\mathcal{S}_n$ to be the sets of vertices internal to a, b-paths. One easily sees that the sets in $\mathcal{S}_n$ correspond to the subsequences of $1, 2, \ldots, n$ that never skip two or more consecutive numbers.



2. The Duality Theory

Let X be a fixed finite set and let $\mathcal{P}(X)$ denote the collection of all subsets of X. A collection of subsets which has the property that no one of its sets is a proper subset of any other is called a *clutter*. One way to convert an arbitrary subcollection of $\mathcal{P}(X)$ into a clutter is to take its minimal sets; this is accomplished by applying the *floor* operator $\lfloor\ \rfloor$: for each $\mathcal{S} \subseteq \mathcal{P}(X)$, $\lfloor \mathcal{S} \rfloor$ is defined to be the minimal subsets in $\mathcal{S}$ (under set inclusion). We start our study by listing the fundamental properties of clutters and the floor operator. These properties are easily verified and are listed without proof.

Lemma 2.1. *Let X and $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(X)$ be given.*

a. *If $\mathcal{S}$ is a clutter and $\mathcal{T} \subseteq \mathcal{S}$, then $\mathcal{T}$ is a clutter.*
b. *$\lfloor \mathcal{S} \rfloor$ is a clutter.*
c. *$\lfloor \mathcal{S} \rfloor \subseteq \mathcal{S}$, with equality if and only if $\mathcal{S}$ is a clutter.*
d. *$\lfloor \lfloor \mathcal{S} \rfloor \rfloor = \lfloor \mathcal{S} \rfloor$.*

A second simple operator mapping $\mathcal{P}(X)$ into $\mathcal{P}(X)$ assigns to a collection, $\mathcal{S} \subseteq \mathcal{P}(X)$, the collection of all sets containing a set in the given collection:

$$\mathcal{S}^s = \{A \in \mathcal{P}(X) : A \supseteq B \text{ for some } B \in \mathcal{S}\}.$$

It is easy to see that this *superset* operator is also idempotent and has a simple interaction with the floor operator.

Lemma 2.2. *Let X and $\mathcal{S}$, $\mathcal{T} \subseteq \mathcal{P}(X)$ be given.*

a. $\mathcal{S} \subseteq S^s$
b. $(\mathcal{S}^s)^s = \mathcal{S}^s$.
c. *If $\mathcal{S} \subseteq \mathcal{T}$, then $\mathcal{S}^s \subseteq \mathcal{T}^s$.*
d. $\lfloor \mathcal{S}^s \rfloor = \lfloor \mathcal{S} \rfloor$.
e. $\lfloor \mathcal{S} \rfloor^s = \mathcal{S}^s$.

The third and last of the elementary operators that we will discuss is the *meet* operator $(\ )^m$, the collection of all sets which *meet* (have a non empty intersection with) each set in the collection to which the operator is being applied:

$$\mathcal{S}^m = \{A \in \mathcal{P}(X) : A \cap B \neq \emptyset, \text{ for each } B \in \mathcal{S}\}.$$

Lemma 2.3. *Let X and $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(X)$ be given.*

a. $(\mathcal{S}^m)^s = \mathcal{S}^m$.
b. *If $\mathcal{S} \subseteq \mathcal{T}$, then $\mathcal{S}^m \supseteq \mathcal{T}^m$.*
c. $\lfloor \mathcal{S} \rfloor^m = \mathcal{S}^m$.
d. $\mathcal{S} \subseteq (\mathcal{S}^m)^m$.
e. $(\mathcal{S}^m)^m = \mathcal{S}^s$.
f. $\lfloor (\mathcal{S}^m)^m \rfloor = \lfloor \mathcal{S} \rfloor$.

Proof. If $A \in \mathcal{S}^m$, then A meets each set in $\mathcal{S}$ and so does any superset B of A; so $(\mathcal{S}^m)^s \subseteq \mathcal{S}^m$. And, by Part a of Lemma 2.2, the reverse inclusion holds. If $A \in \mathcal{T}^m$, A meets each set in $\mathcal{T}$. But, since $\mathcal{S} \subseteq \mathcal{T}$, A meets each set in $\mathcal{S}$ and Part b is proved. Applying the meet operator to both sides of the inclusion in Lemma 2.1, Part c and using what we have just proved, we have $\mathcal{S}^m \subseteq \lfloor \mathcal{S} \rfloor^m$. Now suppose that $A \in \lfloor \mathcal{S} \rfloor^m$ and $B \in \mathcal{S}$. Then A meets every set in $\lfloor \mathcal{S} \rfloor$ and B contains some set in $\lfloor \mathcal{S} \rfloor$. Thus A meets B and we conclude that A meets each set $B \in \mathcal{S}$. But then $A \in \mathcal{S}^m$, completing the proof of Part c.

Suppose that $C \in \mathcal{S}$. Then each set in $\mathcal{S}^m$ meets C and, therefore, $C \in (\mathcal{S}^m)^m$, proving the inclusion in Part d. Applying the superset operator to both sides of this inclusion, we have $\mathcal{S}^s \subseteq ((\mathcal{S}^m)^m)^s = (\mathcal{S}^m)^m$ (the equality following from Part a). To show the reverse inclusion, assume that $C \notin \mathcal{S}^s$, i.e., that C contains no set from $\mathcal{S}$. Then $X - C$ meets every set in $\mathcal{S}$ and hence belongs to $\mathcal{S}^m$. Since $(X - C) \cap C = \emptyset$,

$C \notin \mathcal{S}^{mm}$. This completes the proof of Part e. Finally, Part f follows at once from Part e of this result and Part d of Lemma 2.2. □

We now turn to the *blocking sets* operator which is our primary interest. It is the composition of two of these simpler operators:

$$\mathcal{S}^b = \lfloor \mathcal{S}^m \rfloor, \quad \text{for all } \mathcal{S} \subseteq \mathcal{P}(X).$$

Theorem 2.1. *Let X and $\mathcal{S} \subseteq \mathcal{P}(X)$ be given, then*

$$(\mathcal{S}^b)^b = \lfloor \mathcal{S} \rfloor$$

and, when $\mathcal{S}$ is a clutter,

$$(\mathcal{S}^b)^b = \mathcal{S}.$$

Proof.

$$(\mathcal{S}^b)^b = \lfloor \lfloor \mathcal{S}^m \rfloor^m \rfloor = \lfloor (\mathcal{S}^m)^m \rfloor = \lfloor \mathcal{S} \rfloor.$$

The first equality is the definition of the duality operator; the second and third follow by applying Lemma 2.3, Parts c and f. The second conclusion of the theorem follows from the first conclusion and Lemma 2.1, Part c. □

It is easy to find $\mathcal{S}_n^b$, the blocking sets for our family of examples. Any consecutive pair of integers $\{i, i+1\}$ meets every a, b-path. If $Y \subseteq X$ is any set that does not contain a pair of consecutive integers, then Y misses the vertex set of some a, b-path, namely $X - Y$. Hence

$$\mathcal{S}_n^b = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}\}.$$

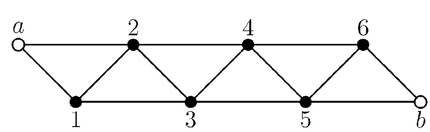
3. The Reliability Polynomial

The *reliability polynomial* of a collection $\mathcal{S}$ of sets in $\mathcal{P}(X)$ is defined by:

$$R_{\mathcal{S}}(p) = \sum_{A \in \mathcal{S}^s} p^{|A|}(1-p)^{|X|-|A|}.$$

If we interpret $E \subseteq X$ to be the result of selecting each element of X with probability p, then $R_{\mathcal{S}}(p)$ is the probability that E will contain a set from $\mathcal{S}$. This is clear once we note that, for each $A \in \mathcal{S}^s$, the corresponding term of $R_{\mathcal{S}}(p)$ is the probability that $E = A$. We point out here that, by Lemma 2.2, Part e, $R_{\lfloor \mathcal{S} \rfloor}(p) = R_{\mathcal{S}}(p)$. Hence, each reliability polynomial is the reliability polynomial of a clutter.

Computing a reliability polynomial can be rather complicated. But, for our examples with n small, the computations are not too difficult. We consider the case $n = 6$:



We easily see that there are no a, b-path, vertex sets with fewer than three internal vertices, and only four with three internal vertices: 1,3,5; 2,3,5; 2,4,5; 2,4,6. These contribute $4p^3(1-p)^3$ to the reliability polynomial. It is also easy to see that all four-sets belong to $\mathcal{S}_6$ except the five complements of consecutive pairs, for $\binom{6}{4} - 5 = 10$ collections, contributing $10p^4(1-p)^2$. Finally, all five-sets and the whole set belong to $\mathcal{S}_6$, contributing $6p^5(1-p)$ and p^6, respectively. Thus

$$R_{\mathcal{S}_6}(p) = 4p^3(1-p)^3 + 10p^4(1-p)^2 + 6p^5(1-p) + p^6 = 4p^3 - 2p^4 - 2p^5 + p^6.$$

It is a bit harder to compute $R_{\mathcal{S}_6^b}(p)$ directly. Instead, we will prove, and then use, the following theorem.

Theorem 3.1. *Let X and $\mathcal{S} \subseteq \mathcal{P}(X)$ be given, then*

$$R_{\mathcal{S}^b}(p) = 1 - R_{\mathcal{S}}(1-p).$$

Proof. Again, we interpret $E \subseteq X$ to be the result of selecting each element of X with probability p. We note that the probability that $X - E = A$ or $E = X - A$ is $p^{|X-A|}(1-p)^{|A|}$. So $R_{\mathcal{S}}(1-p)$ represents the probability that the complement of E contains a set from $\mathcal{S}$ or that E misses some set in $\mathcal{S}$. Now the probability that E contains a set from $\mathcal{S}^b$ is exactly the probability that E meets every set from $(\mathcal{S}^b)^b = \lfloor \mathcal{S} \rfloor$. But the probability that E meets every set from $\lfloor \mathcal{S} \rfloor$ or from $\mathcal{S}$ is 1 minus the probability that E misses some set from $\mathcal{S}$. □

Applying this result to $\mathcal{S}_6^b$, we have:

$$\begin{aligned} R_{\mathcal{S}_6^b}(p) &= 1 - 4(1-p)^3 - 2(1-p)^4 - 2(1-p)^5 + (1-p)^6 \\ &= 5p^2 - 4p^3 - 3p^4 + 4p^5 - p^6. \end{aligned}$$

4. The Inclusion–Exclusion Polynomial

Let X be a finite set and let $\mathcal{S} \subseteq \mathcal{P}(X)$. We associate the following polynomial with $\mathcal{S}$:

$$I_{\mathcal{S}}(p) = \sum_{\emptyset \subset \mathcal{U} \subseteq \mathcal{S}} (-1)^{(|\mathcal{U}|-1)} p^{|\bigcup_{A \in \mathcal{U}} A|}.$$

Later we will show that this is none other than the reliability polynomial for $\mathcal{S}$. However, as the terms in this sum and those in the sum defining the reliability polynomial do not match, the equality is not at all obvious. So, for now, we call $I_{\mathcal{S}}(p)$ the *inclusion–exclusion polynomial* of the collection $\mathcal{S}$. In this form, the polynomial was defined by the second author as a generating function for the coefficients of some inclusion–exclusion sums that arise in sampling. In these applications, the variable p is not interpreted as a probability and the polynomials $I_{\mathcal{S}}(p)$ and $I_{\mathcal{S}^b}(p)$ have no probabilistic significance as functions. But their coefficients enable one to compute probabilities which depend on the structure of the collection $\mathcal{S}$ in terms of simple probabilities which depend only on the method of sampling.

Select a method of sampling from the n-set X and let P$[h : k]$ denote the probability that an arbitrary k-sample will miss (have an empty intersection with) a fixed h-subset. If we select k elements from X without replacement, then $\mathrm{P}[h:k] = \frac{\binom{n-h}{k}}{\binom{n}{k}}$; if we select with replacement, then $\mathrm{P}[h:k] = (\frac{n-h}{n})^k$. Now fix the sample size k. For $A \in \mathcal{S}$, let E_A denote the event: "the selected k-sample is disjoint from A," that is, E_A is the collection of all k-samples that are disjoint from A. The union $\bigcup_{A\in\mathcal{S}} E_A$ is then the event: "the selected k-sample is disjoint from some set in $\mathcal{S}$" and $\bigcap_{A\in\mathcal{U}} E_A$ denotes the event: "the selected k-sample is disjoint from all of the sets in the collection $\mathcal{U}$." By the inclusion–exclusion formula for probabilities, we have:

$$\mathrm{P}\left[\bigcup_{A\in\mathcal{S}} E_A\right] = \sum_{\emptyset\subset\mathcal{U}\subseteq\mathcal{S}} (-1)^{(|\mathcal{U}|-1)}\mathrm{P}\left[\bigcap_{A\in\mathcal{U}} E_A\right].$$

Next we observe that $\mathrm{P}[\bigcap_{A\in\mathcal{U}} E_A]$ is the probability that the k-sample is disjoint from the set $\bigcup_{A\in\mathcal{U}} A$. Hence, $\mathrm{P}[\bigcap_{A\in\mathcal{U}} E_A] = P[|\bigcup_{A\in\mathcal{U}} A| : k]$ and

$$\mathrm{P}\left[\bigcup_{A\in\mathcal{S}} E_A\right] = \sum_{\emptyset\subset\mathcal{U}\subseteq\mathcal{S}} (-1)^{(|\mathcal{U}|-1)}\mathrm{P}\left[\left|\bigcup_{A\in\mathcal{U}} A\right| : k\right].$$

Thus, the coefficient of $\mathrm{P}[h:k]$ in this sum is the same as the coefficient of p^h in the inclusion–exclusion polynomial. We have proved the first part of Theorem 4.1.

Theorem 4.1. *Let the n-set X, the clutter $\mathcal{S} \subseteq \mathcal{P}(X)$ and a method of sampling from X be given. Let D_k denote the event "the selected k-sample is disjoint from some set in $\mathcal{S}$" and C_k denote the event "the selected k-sample contains no set from $\mathcal{S}$". Finally, let $I_{\mathcal{S}}(p) = \sum_{i=1}^n a_i p^i$ and let $I_{\mathcal{S}^b}(p) = \sum_{i=1}^n b_i p^i$. Then*:

a. $P[D_k] = \sum_{h=1}^n a_h P[h:k]$;
b. $P[C_k] = \sum_{h=1}^n b_h P[h:k]$.

Proof. To prove Part b, we need only observe that a subset of X will be disjoint from some set in $\mathcal{S}^b$ if and only if it contains no subset from $\mathcal{S}$. □

The following corollary is immediate.

Corollary 4.1. *Let the n-set X, the clutter $\mathcal{S} \subseteq \mathcal{P}(X)$ and a method of sampling from X be given and let $I_{\mathcal{S}}(p) = \sum_{i=1}^n a_i p^i$ and $I_{\mathcal{S}^b}(p) = \sum_{i=1}^n b_i p^i$. Then*:

a. *The probability that a k-sample meets every set in $\mathcal{S}$ is* $1 - \sum_{h=1}^n a_h P[h:k]$;
b. *The probability that a k-sample contains some set in $\mathcal{S}$ is* $1 - \sum_{h=1}^n b_h P[h:k]$.

Computing the inclusion–exclusion polynomial directly from its definition is quite different from the computation of the reliability polynomial as defined above. Since this computational method is the basis of a later application, we illustrate it by computing $I_{\mathcal{S}_6}(p)$ and $I_{\mathcal{S}_6^b}(p)$. Here $I_{\mathcal{S}_6^b}(p)$ is the easier to compute. To do this, we must investigate the $2^5 - 1$ unions of non empty subcollections from $\mathcal{S}_6^b$. The easiest way to keep track of all of these terms is to first construct the matrix $\mathbf{M}$ where the entry $\mathbf{M}_{ij}$ is the number of subcollections consisting of i sets from $\mathcal{S}_6^b$ that have a union of size j. Note that $\mathbf{M}$ is an $m \times n$ matrix where $m = |\mathcal{S}_6^b| = 5$ and $n = |X| = 6$. Once we have this matrix, the coefficient a_h of p^h in $I_{\mathcal{S}}(p)$ is given by $a_h = \sum_{i=1}^m (-1)^{i-1}\mathbf{M}_{ih}$.

Since all five of our sets have two elements, $\mathbf{M}_{1,2} = 5$ and all other entries in the first row are zero. The union of two blocking sets contains either three elements ($i, i+1, i+2$; 4 cases) or four elements (disjoint blocking sets; six cases). This accounts for all $\binom{5}{2}$ pairs of blocking sets and gives $\mathbf{M}_{2,3} = 4$ and $\mathbf{M}_{2,4} = 6$; the remaining entries in row 2 must then be 0. Of the ten collections of three blocking sets, three collections have a union of size four and one collection has a union of size six, so, the remaining six collections of three blocking sets must have unions of size five. Just two of the $\binom{5}{4}$ collections of four blocking sets have union size five, the remaining three have union size six as does the entire collection. The results are tabulated in the matrix $\mathbf{M}$, below:

$$\mathbf{M} = \begin{bmatrix} 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 6 & 0 & 0 \\ 0 & 0 & 0 & 3 & 6 & 1 \\ 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus we have:

$$I_{\mathcal{S}_6^b}(p) = 5p^2 - 4p^3 + (-6+3)p^4 + (6-2)p^5 + (1-3+1)p^6 = R_{\mathcal{S}_6^b}(p),$$

as expected.

We should note that the duality formula $I_{\mathcal{S}^b}(p) = 1 - I_{\mathcal{S}}(1-p)$, that was so easy to prove for the reliability polynomial, is quite difficult to prove using this inclusion–exclusion definition. Hence, instead of trying to prove that result for the inclusion–exclusion polynomial, we move directly to the proof of the equality of these two polynomials.

5. The Equivalance Theorem

We start this section by introducing two constructions that are essential to the induction proof of the main result. Let $\mathcal{S} \subseteq \mathcal{P}(X)$ and let $Y \subset X$. There are two collections of subsets of Y which one may naturally associate with $\mathcal{S}$. They are called the *restriction* of $\mathcal{S}$ to Y, $\rho_Y(\mathcal{S})$, and the *projection* of $\mathcal{S}$ onto Y, $\pi_Y(\mathcal{S})$, and they are defined by:

$$\rho_Y(\mathcal{S}) = \{A : A \in \mathcal{S}, A \subseteq Y\} \quad \text{and} \quad \pi_Y(\mathcal{S}) = \{A \cap Y : A \in \mathcal{S}\}.$$

Lemma 5.1. *Let X, $y \in X$ and $\mathcal{S} \subseteq \mathcal{P}(X)$ be given; let $Y = X - \{y\}$. Then:*

$$I_{\mathcal{S}}(p) = (1-p)I_{\rho_Y(\mathcal{S})}(p) + pI_{\pi_Y(\mathcal{S})}(p).$$

Proof. Let $\mathbf{Q} = \{\mathcal{U} : \emptyset \subset \mathcal{U} \subseteq \mathcal{S}\}$ and partition $\mathbf{Q}$ into cells $\mathbf{Q}_1$ and $\mathbf{Q}_2$ where:

$$\mathbf{Q}_1 = \left\{\mathcal{U} : y \notin \bigcup_{A \in \mathcal{U}} A\right\} \quad \text{and} \quad \mathbf{Q}_2 = \left\{\mathcal{U} : y \in \bigcup_{A \in \mathcal{U}} A\right\}.$$

Next let

$$g_i(p) = \sum_{\mathcal{U} \in \mathbf{Q}_i} (-1)^{(|\mathcal{U}|-1)} p^{|\bigcup_{A \in \mathcal{U}} A|}, \quad \text{for } i = 1, 2.$$

We claim that:

$$I_{\mathscr{S}}(p) = g_1(p) + g_2(p), \tag{1}$$

$$g_1(p) = I_{\rho_Y(\mathscr{S})}(p) \quad \text{and} \tag{2}$$

$$g_2(p) = p[I_{\pi_Y(\mathscr{S})}(p) - I_{\rho_Y(\mathscr{S})}(p)]. \tag{3}$$

The first two equalities are trivial. To verify the third, consider $\mathscr{U} \subseteq \mathscr{S}$.

- Assume first that $y \in \bigcup_{A\in\mathscr{U}} A$. Then $\mathscr{U}$ contributes $(-1)^{(|\mathscr{U}|-1)}p^{|\bigcup_{A\in\mathscr{U}} A|}$ to $g_2(p)$. In addition, $\mathscr{U}$ contributes $(-1)^{(|\mathscr{U}|-1)}p^{|\bigcup_{A\in\mathscr{U}} A|-1}$ to $I_{\pi_Y(\mathscr{S})}(p)$ and nothing to $I_{\rho_Y(\mathscr{S})}(p)$; hence, $\mathscr{U}$ also contributes $(-1)^{(|\mathscr{U}|-1)}p^{|\bigcup_{A\in\mathscr{U}} A|}$ to $p[I_{\pi_Y(\mathscr{S})}(p) - I_{\rho_Y(\mathscr{S})}(p)]$.
- Next assume that $y \notin \bigcup_{A\in\mathscr{U}} A$. Then $\mathscr{U}$ contributes nothing to $g_2(p)$. In addition, $\mathscr{U}$ contributes $(-1)^{(|\mathscr{U}|-1)}p^{|\bigcup_{A\in\mathscr{U}} A|}$ to both $I_{\pi_Y(\mathscr{S})}(p)$ and $I_{\rho_Y(\mathscr{S})}(p)$; hence $\mathscr{U}$ also contributes nothing to $p[I_{\pi_Y(\mathscr{S})}(p) - I_{\rho_Y(\mathscr{S})}(p)]$.

Equation (3) follows at once. Substituting (2) and (3) into (1) gives the desired formula for $I_{\mathscr{S}}(p)$. □

The last step before proving that $I_{\mathscr{S}}(p)$ and $R_{\mathscr{S}}(p)$ are the same is to show that the reliability polynomial also satisfies this decomposition equation.

Lemma 5.2. *Let X, $y \in X$ and $\mathscr{S} \subseteq \mathscr{P}(X)$ be given; let $Y = X - y$. Then:*

$$R_{\mathscr{S}}(p) = (1-p)R_{\rho_Y(\mathscr{S})}(p) + pR_{\pi_Y(\mathscr{S})}(p).$$

Proof. Let $\mathscr{N}_i$ denote the collection of all i-element subsets that belong to $\mathscr{S}^s$. We partition $\mathscr{N}_i$ into $\mathscr{M}_i$, those sets in $\mathscr{N}_i$ that do not contain y, and $\mathscr{K}_i$, those that do contain y. We have:

$$R_{\rho_Y(\mathscr{S})}(p) = \sum_{i=0}^{|X|-1} |\mathscr{M}_i| p^i (1-p)^{|X|-1-i} \quad \text{so,}$$

$$(1-p)R_{\rho_Y(\mathscr{S})}(p) = \sum_{i=0}^{|X|-1} |\mathscr{M}_i| p^i (1-p)^{|X|-i};$$

and

$$R_{\pi_Y(\mathscr{S})}(p) = \sum_{j=0}^{|X|-1} |\mathscr{K}_{j+1}| p^j (1-p)^{|X|-1-j} \quad \text{so,}$$

$$pR_{\pi_Y(\mathscr{S})}(p) = \sum_{j=0}^{|X|-1} |\mathscr{K}_{j+1}| p^{j+1} (1-p)^{|X|-(1+j)} = \sum_{i=1}^{|X|} |\mathscr{K}_i| p^i (1-p)^{|X|-i}.$$

Thus,

$$(1-p)R_{\rho_Y(\mathscr{S})}(p) + pR_{\pi_Y(\mathscr{S})}(p) = \sum_{i=0}^{|X|-1} |\mathscr{M}_i| p^i (1-p)^{|X|-i} + \sum_{i=1}^{|X|} |\mathscr{K}_i| p^i (1-p)^{|X|-i}$$

$$= \sum_{i=0}^{|X|} |\mathscr{N}_i| p^i (1-p)^{|X|-i} = R_{\mathscr{S}}(p).$$

□

Theorem 5.1. *Let X and $\mathcal{S} \subseteq \mathcal{P}(X)$ be given. Then*:

$$I_{\mathcal{S}}(p) = R_{\mathcal{S}}(p).$$

Proof. We proceed by induction on $|X|$. If $|X| = 1$, there are four rather trivial cases, for which one easily verifies the equality:

$$\mathcal{S} = \emptyset,\ \ I_{\mathcal{S}}(p) = R_{\mathcal{S}}(p) = 0;$$
$$\mathcal{S} = \emptyset,\ \ I_{\mathcal{S}}(p) = R_{\mathcal{S}}(p) = 1;$$
$$\mathcal{S} = \{\emptyset, X\},\ \ I_{\mathcal{S}}(p) = R_{\mathcal{S}}(p) = 1 \quad \text{and}$$
$$\mathcal{S} = X,\ \ I_{\mathcal{S}}(p) = R_{\mathcal{S}}(p) = p.$$

For $|X| > 1$, fix $y \in X$ and let $Y = X - y$. We have:

$$\begin{aligned} I_{\mathcal{S}}(p) &= (1-p)I_{\rho_Y(\mathcal{S})}(p) + pI_{\pi_Y(\mathcal{S})}(p) \\ &= (1-p)R_{\rho_Y(\mathcal{S})}(p) + pR_{\pi_Y(\mathcal{S})}(p) = R_{\mathcal{S}}(p), \end{aligned}$$

where the first equality holds by Lemma 5.1, the second by the induction hypothesis, and the third by Lemma 5.2. □

This result had been anticipated in so far as inclusion–exclusion arguments have been used to compute some reliability polynomials. See Colbourn (1993).

6. Applying the Reliability – Inclusion/Exclusion Polynomial

We will illustrate the uses of the reliability polynomial with our set of examples $\mathcal{S}_n$ and $\mathcal{S}_n^b$ as described in the introduction. Our first task is to compute $R_{\mathcal{S}_n}(p)$. This is most easily done by showing that $R_{\mathcal{S}_n}(p)$ satisfies the recursion formula

$$R_{\mathcal{S}_n}(p) = pR_{\mathcal{S}_{n-1}}(p) + p(1-p)R_{\mathcal{S}_{n-2}}(p), \quad \text{with } R_{\mathcal{S}_0}(p) = R_{\mathcal{S}_1}(p) = 1.$$

To prove this, we partition the subsets of $X = \{1, 2, \ldots, n\}$ that belong to $\mathcal{S}_n$ into those that end in n and those that end in $(n-1)$. Those that end in n are all obtained by adding n to a set that contributes $p^k(1-p)^{(n-1)-k}$ to $R_{\mathcal{S}_{n-1}}(p)$, for some k. The amended set then contributes $p^{k+1}(1-p)^{n-(k+1)}$ to $R_{\mathcal{S}_n}(p)$. Combining all of these terms gives $pR_{\mathcal{S}_{n-1}}(p)$. Those sets that end in $(n-1)$ are all obtained by adding $(n-1)$ to a set that contributes $p^k(1-p)^{(n-2)-k}$ to $R_{\mathcal{S}_{n-2}}(p)$, for some k. The amended set then contributes $p^{k+1}(1-p)^{n-(k+1)}$ to $R_{\mathcal{S}_n}(p)$. Combining all of these terms gives $p(1-p)R_{\mathcal{S}_{n-2}}(p)$. Thus

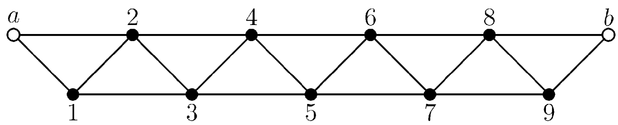
$$R_{\mathcal{S}_n}(p) = pR_{\mathcal{S}_{n-1}}(p) + p(1-p)R_{\mathcal{S}_{n-2}}(p).$$

Computing $R_{\mathcal{S}_2}(p) = 2p - p^2$ and $R_{\mathcal{S}_3}(p) = p + p^2 - p^3$ directly verifies that the definitions $R_{\mathcal{S}_0}(p) = R_{\mathcal{S}_1}(p) = 1$ are consistent with the recursion.

This is a linear recursion with characteristic polynomial $\alpha^2 - p\alpha + p(1-p)$. Hence, $R_{\mathcal{S}_n}(p)$ is a linear combination of the nth powers of the roots of this polynomial. Using the initial conditions to compute the coefficients, we have, in closed form

$$R_{\mathscr{S}_n}(p) = \left(\frac{1}{2} + \frac{2-p}{2\sqrt{4p-3p^2}}\right)\left(\frac{p+\sqrt{4p-3p^2}}{2}\right)^n + \left(\frac{1}{2} - \frac{2-p}{2\sqrt{4p-3p^2}}\right)\left(\frac{p-\sqrt{4p-3p^2}}{2}\right)^n.$$

Now lets return to our initial example $\mathscr{S}_9$:

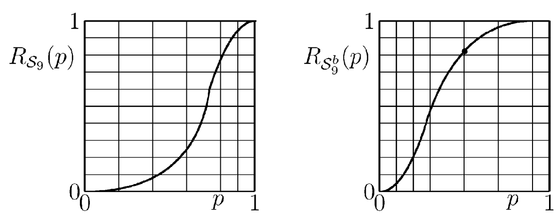


We have

$$R_{\mathscr{S}_9}(p) = p^4 + 10p^5 - 15p^6 + 3p^7 + 3p^8 - p^9 \quad \text{and}$$

$$R_{\mathscr{S}_9^b}(p) = 8p^2 - 7p^3 - 15p^4 + 25p^5 - 6p^6 - 9p^7 + 6p^8 - p^9.$$

Let Y denote the subset of $X = \{1, \ldots, 9\}$ obtained by including each of the integers in X with probability p. Then $R_{\mathscr{S}_9}(p)$ is the probability that Y is the set of internal vertices of an a, b-path. $R_{\mathscr{S}_9^b}(p)$ is the probability that Y is a blocking set, that is, that $X - Y$ is not the set of internal vertices of an a, b-path. An alternative interpretation of $R_{\mathscr{S}_9^b}(p)$ is the probability that Y contains two consecutive integers from X. The best way to exhibit this interpretation of the reliability polynomial is to graph these functions. From the right-hand graph we see that, when each integer from 1 to 9 is included with probability $\frac{1}{2}$, the resulting collection will contain two consecutive integers with a probability approximately 0.82.



With the inclusion–exclusion polynomial interpretation at our disposal, we may say much more. Using part b of Corollary 4.1 to Theorem 4.1, we have that the probability that a randomly selected k-sample will contain the vertices of an a, b-path is:

$$1 - \frac{8\binom{7}{k} - 7\binom{6}{k} - 15\binom{5}{k} + 25\binom{4}{k} - 6\binom{3}{k} - 9\binom{2}{k} + 6\binom{1}{k} - \binom{0}{k}}{\binom{9}{k}},$$

when sampling without replacement, and

$$1-\frac{8\times 7^k-7\times 6^k-15\times 5^k+25\times 4^k-6\times 3^k-9\times 2^k+6}{9^k},$$

when sampling with replacement. We summarize these computations in the below table:

The probability that a k-sample contains the vertices of an a, b-path in $\mathscr{S}_9$

Sampling\\k	1	2	3	4	5	6	7	8	9	10	11	12
Without rep.	0	0	0	.0079	.1190	.4167	.7778	1	1	–	–	–
With rep.	0	0	0	.0037	.0345	.1012	.1946	.3008	.4078	.5079	.5969	.6734

Similarly, using $R_{\mathscr{S}_9}(p)=p^4+10p^5-15p^6+3p^7+3p^8-p^9$, we have the following table:

The probability that a k-sample contains two consecutive integers from $\{1,\ldots,9\}$

Sampling\\k	1	2	3	4	5	6	7	8	9	10	11	12
Without rep.	0	.2222	.5833	.8810	.9921	1	1	1	1	–	–	–
With rep.	0	.1975	.4691	.6920	.8337	.9137	.9562	.9780	.9890	.9944	.9972	.9986

Because of the linearity of the equations in Theorem 4.1, several other probabilistic formulas may be derived. We illustrate this with expected wait time. Consider selecting a sample one element at a time, with or without replacement. We denote the expected wait time for the sample to meet all subsets in $\mathscr{S}$ (the expected size of a minimal sample that meets all sets in $\mathscr{S}$ or the expected size of a minimal sample that contains a blocking set for $\mathscr{S}$) by $\mathrm{E}_{\mathscr{S}^b}$. Recall that D_k denotes the event "the selected k-sample is disjoint from some set in $\mathscr{S}$." Thus, the probability that the selected k-sample contains some set in $\mathscr{S}^b$ is $1-\mathrm{P}[D_k]$. We have:

$$\mathrm{E}_{\mathscr{S}^b}=\sum_{k=1}^{\infty}k\times(\mathrm{P}[D_{k-1}]-\mathrm{P}[D_k])=\sum_{k=0}^{\infty}\mathrm{P}[D_k].$$

Now if $R_{\mathscr{S}}(p)=\sum_{i=1}^{n}a_ip^i$, $\mathrm{P}[D_k]=\sum_{h=1}^{n}a_h\mathrm{P}[h:k]$ and so:

$$\mathrm{E}_{\mathscr{S}^b}=\sum_{k=0}^{\infty}\sum_{h=1}^{n}a_h\mathrm{P}[h:k]=\sum_{h=1}^{n}a_h\sum_{k=0}^{\infty}\mathrm{P}[h:k].$$

One easily computes $\sum_{k=0}^{\infty}\mathrm{P}[h:k]$. (the expected waiting time for a sample to meet a fixed h-set) to be $\frac{n+1}{h+1}$, when sampling without replacement, and $\frac{n}{h}$, when sampling with replacement. We have proved the first part of the following corollary, the second part follows by a dual argument.

Corollary 6.1. *Let the set X, the clutter $\mathscr{S}\subseteq P(X)$ and a method of sampling from X be given. Let $R_{\mathscr{S}}(p)=\sum_{i=1}^{n}a_ip^i$ and let $R_{\mathscr{S}^b}(p)=\sum_{i=1}^{n}b_ip^i$.*

a. $$\mathrm{E}_{\mathscr{S}^b}=\begin{cases}(n+1)\sum_{h=1}^{n}\frac{a_h}{h+1}, & \textit{without replacement};\\ (n)\sum_{h=1}^{n}\frac{a_h}{h}, & \textit{with replacement.}\end{cases}$$

b. $E_{\mathcal{S}} = \begin{cases} (n+1)\sum_{h=1}^{n} \frac{b_h}{h+1}, & \textit{without replacement}; \\ (n)\sum_{h=1}^{n} \frac{b_h}{h}, & \textit{with replacement.} \end{cases}$

Applying this corollary to our example, we have that the expected wait time for a sample from $X = \{1, \dots, 9\}$ to contain a pair of consecutive integers is

$$10\left(\frac{1}{5} + \frac{10}{6} - \frac{15}{7} + \frac{3}{8} + \frac{3}{9} - \frac{1}{10}\right) \approx 3.321, \quad \text{when sampling without replacement;}$$

$$9\left(\frac{1}{4} + \frac{10}{5} - \frac{15}{6} + \frac{3}{7} + \frac{3}{8} - \frac{1}{9}\right) \approx 3.982, \quad \text{when sampling with replacement;}$$

and the expected wait time for a sample to meet every pair of consecutive integers from X (to contain the vertices of an a, b-path) is

$$10\left(\frac{8}{3} - \frac{7}{4} - \frac{15}{5} + \frac{25}{6} - \frac{6}{7} - \frac{9}{8} + \frac{6}{9} - \frac{1}{10}\right) \approx 6.679,$$

when sampling without replacement;

$$9\left(\frac{8}{2} - \frac{7}{3} - \frac{15}{4} + \frac{25}{5} - \frac{6}{6} - \frac{9}{7} + \frac{6}{8} - \frac{1}{9}\right) \approx 11.429,$$

when sampling with replacement.

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