

## A note on natural correspondences that satisfy exclusion

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Let  $X$  be a finite set with  $m$  elements. A function  $F : \mathcal{P}_2(X) \rightarrow \mathcal{P}(X)$  (from the collection of all 2-element subsets of  $X$  into the collection of all subsets of  $X$ ) is called a *natural correspondence* if  $S \subseteq F(S)$ , for all  $S \in \mathcal{P}_2(X)$ . We say that  $F$  satisfies *exclusion* if there exists some 3-element subset  $\{x, y, z\}$  such that  $x \notin F(\{y, z\})$ ,  $y \notin F(\{x, z\})$  and  $z \notin F(\{x, y\})$ .

**Observation 1**  $F$  does not satisfy exclusion if and only if, for every  $T \in \mathcal{P}_3(X)$ , there exists some  $S \in \mathcal{P}_2(X)$  such that  $S \subseteq T \subseteq F(S)$ .

Let  $F : \mathcal{P}_2(X) \rightarrow \mathcal{P}(X)$  be a natural correspondence; we define the average image size  $a_F$  by:

$$a_F = \frac{1}{\binom{m}{2}} \sum_{S \in \mathcal{P}_2(X)} |F(S)|$$

**Theorem 1** Let  $F : \mathcal{P}_2(X) \rightarrow \mathcal{P}(X)$  be a natural correspondence that does not satisfy exclusion. Then  $a_F \geq \frac{m+4}{3}$ .

*Proof* Since  $F$  does not satisfy exclusion, we may assign to each  $T \in \mathcal{P}_3(X)$  a subset  $S \in \mathcal{P}_2(X)$  such that  $S \subseteq T \subseteq F(S)$ . Now consider a subset  $S \in \mathcal{P}_2(X)$  and let  $T_1, \dots, T_k$  be the 3-element subsets to which  $S$  has been assigned. Then  $|F(S)| \geq k + 2$ . Summing this inequality over all 2-element subsets gives:

$$\sum_{S \in \mathcal{P}_2(X)} |F(S)| \geq \binom{m}{3} + 2 \binom{m}{2}.$$

Dividing through by  $\binom{m}{2}$  and simplifying gives the result. □

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**Corollary 1** *If  $F : \mathcal{P}_2(X) \rightarrow \mathcal{P}(X)$  is a natural correspondence with  $a_F < \frac{m+4}{3}$ , then  $F$  satisfies exclusion.*

This result is *best possible* in that, for any  $m$ , one may construct a natural correspondence  $F : \mathcal{P}_2(X) \rightarrow \mathcal{P}(X)$  that does not satisfy exclusion for which  $a_F = \frac{m+4}{3}$ . First, to each  $T \in \mathcal{P}_3(X)$ , assign a subset  $S \in \mathcal{P}_2(X)$  such that  $S \subseteq T$ . Now, for each subset  $S \in \mathcal{P}_2(X)$ , let  $T_1, \dots, T_k$  be the collection of 3-element subsets to which  $S$  has been assigned and define  $F(S) = T_1 \cup \dots \cup T_k \cup S$ . We close with a simple example.

*Example 1* Let  $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ . We will denote our subsets by their characteristic functions: 11010000 denotes  $\{1, 2, 4\}$ ; in fact since our example will be symmetric under cyclic permutations, 11010000 will denote the sets  $\{1, 2, 4\}, \{2, 3, 5\}, \dots, \{8, 1, 3\}$ .

Up to a cyclic permutation, there are only 4 types of 2-element subsets:  $\alpha$ , 11000000;  $\beta$ , 10100000;  $\gamma$ , 10010000;  $\delta$ , 10001000 and 7 types of 3-element subsets:  $a$ , 11100000;  $b$ , 11010000;  $c$ , 11001000;  $d$ , 11000100;  $e$ , 11000010;  $f$ , 10101000;  $g$ , 10100100. To each 3-subset of type  $a$  or  $b$  assign the 2-subset of type  $\beta$  that it contains; to each 3-subset of type  $c$  or  $g$  assign the 2-subset of type  $\gamma$  that it contains; to each 3-subset of type  $d$  or  $e$  assign the 2-subset of type  $\alpha$  that it contains; finally to each 3-subset of type  $f$  assign the 2-subset of type  $\delta$  that it contains. Then

$$\begin{aligned} F(11000000) &= 11000110, & F(10100000) &= 11100001, \\ F(10010000) &= 10010011, & F(10001000) &= 10101010, \end{aligned}$$

and  $a_F = 4 = \frac{8+4}{3}$ .

**Reference**

CK Campbell DE, Kelly JS (2004) Social welfare functions that satisfy Pareto, anonymity and neutrality, but not IIA