DESIGNING A MIRROR THAT INVERTS IN A CIRCLE

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Dedicated to our mentors
George Piranian, Ernst Snapper, and Max Schiffer

1. Introduction

If $\mathcal{C}$ is a circle with center O and P is a point distinct from O in the plane of $\mathcal{C}$, the inverse (image) of P under inversion in $\mathcal{C}$ is the unique point Q on the ray from O through P so that the product of the lengths of the segments $\overline{OQ}$ and $\overline{OP}$ is equal to the square of the radius of $\mathcal{C}$. Like reflection in a line, inversion in a circle can be easily carried out pointwise with a straightedge and a pair of compasses.

During the early part of the industrial revolution, engineers and mathematicians tried to design linkages to carry out these transformations. Linkages for reflection in a line were easy to produce. The interest in the more difficult problem of designing a linkage for inversion in a circle $\mathcal{C}$ is based on the well-known fact that, under inversion in $\mathcal{C}$, circles through O become lines not through O and lines not through O become circles through O. In 1864 the French military engineer Peaucellier designed a linkage that converts circular motion to mathematically perfect linear motion. Cf. [1; Ch. 4] and [2].

Since reflection in a line can be effected with a flat mirror while controlled optical distortions can be produced through reflection (in the optical sense) in curved mirrors, it is natural to wonder whether inversion in a circle can be achieved through reflection in a suitable mirrored surface. In this note we give some positive answers to this question, including equations for constructing such mirrors. Specifically, we show how to design a mirror in which the viewer sees the exterior of a disk as though it had been geometrically inverted to the interior of the disk.

1.1. **The Mirror.**

If such a mirror exists, it is a surface of revolution somewhat similar in shape to a cone. (In fact, it more closely resembles a bell.) Its exact shape depends upon the point E where the observer's eye is located on the axis of revolution, which we take to be the y-axis

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of a standard euclidean coordinate system in $\mathbf{R}^3$. We further suppose that E is above the xz-plane which meets the mirror in a circle of radius $r_0 \leq 1$ centered at the origin.

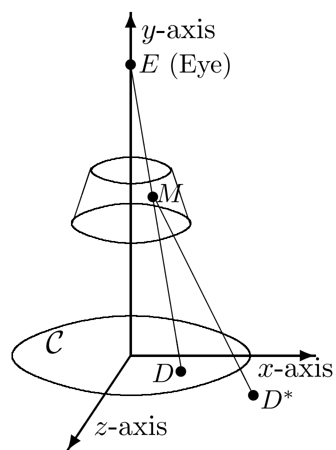

Figure 1

Under simple optical inversion with respect to the unit circle $\mathcal{C}$ in the xz-plane, a dot at a point D^* in the plane outside $\mathcal{C}$ would be seen by the observer at E as if it were located inside $\mathcal{C}$ at the point D on the segment between the origin O and D^* for which $|\overline{OD^*}| \cdot |\overline{OD}| = 1$. To achieve this, our mirror must reflect a ray from D^* to E at an intermediate point M in such a way that the reflected ray appears to come from D, as indicated in Figure 1. (From geometric optics, the tangent line to the mirror surface at M in the plane containing the incident ray and the reflected ray makes equal angles with these rays.) The mirror images of lines outside $\mathcal{C}$ would then appear as circles inside $\mathcal{C}$.

It will suffice to restrict our attention to a tangent line to the cross section of the mirror in the xy-plane, as depicted in Figure 2. In this figure, Y is the y-coordinate of the point E (the observer's eye), w^* is the x-coordinate of the point on the x-axis whose reflection is being viewed by the observer, and w is is the x-coordinate of its virtual image.

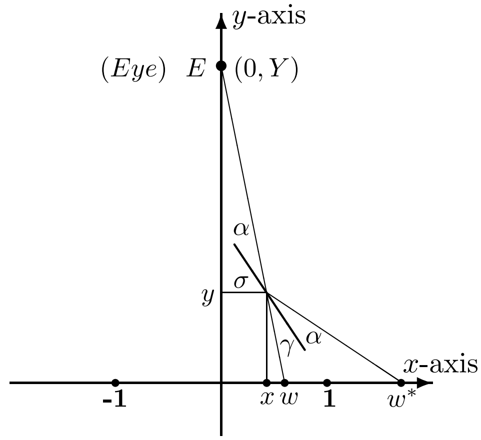

Figure 2

1.2. **The Differential Equation.** Let $y = f(x)$ be the equation of the cross section of the hypothesized mirror for $x \geq 0$. If (x, y) represents a point on the mirror, let α denote the angle that the tangent line to the graph of f at (x, y) makes with the line of sight from the observer at $(0, Y)$ to this point. Let σ denote the angle the tangent line makes with the horizontal and γ the angle it makes with the vertical. We note that $\frac{dx}{dy} = -\tan(\gamma)$ and conclude that

$$\tan(\gamma) = \frac{-1}{y'}. \tag{1}$$

There are four other relations that we can easily see from Figure 2:

$$w^* = \frac{Y - y}{xY}; \tag{2}$$

$$\sigma + \gamma = \frac{\pi}{2}; \tag{3}$$

$$\tan(\alpha + \sigma) = \frac{Y - y}{x}; \tag{4}$$

$$\tan(\alpha + \gamma) = \frac{w^* - x}{y}. \tag{5}$$

From (1) we get

$$u \doteq -\tan(2\gamma - \frac{\pi}{2}) = \frac{1}{\tan(2\gamma)} = \frac{1 - \tan^2(\gamma)}{2\tan(\gamma)} = \frac{1 - (y')^2}{2y'};$$

also by (2) - (5)

$$\begin{aligned}
\tan(2\gamma - \frac{\pi}{2}) &= \tan(\gamma - \sigma) = \tan((\alpha + \gamma) - (\alpha + \sigma)) \\
&= \frac{\frac{w^* - x}{y} - \frac{Y - y}{x}}{1 + \left(\frac{w^* - x}{y}\right)\left(\frac{Y - y}{x}\right)} \\
&= \frac{x(w^* - x) - y(Y - y)}{xy + (w^* - x)(Y - y)} \\
&= x\frac{(1 - Yy)(Y - y) - x^2 Y}{x^2 Y y + (Y - y)(Y - y - x^2 Y)}
\end{aligned}$$

so that

$$u = x\frac{(Y - y)(yY - 1) + x^2 Y}{(Y - y)^2(1 - x^2) + x^2 y^2}. \tag{6}$$

The first expression for u gives the quadratic equation $(y')^2 + 2uy' - 1 = 0$. Noting that y' is never positive, we see that

$$y' = -u - \sqrt{u^2 + 1}; \tag{7}$$

and when (6) is used to replace u, we get a first-order differential equation for the meridian curve. Note that $y' = -1$ when $x = 0$.

Before working with this general equation, we consider the more tractable limiting case as the viewer moves toward positive infinity.

2. The View from Infinity

When $Y \to \infty$, we see from (6) that $u \to \frac{xy}{1-x^2}$; and, when $u = \frac{xy}{1-x^2}$, the right side of (7) has the partial derivative with respect to y given by

$$-\left(1 + \frac{u}{\sqrt{1 + u^2}}\right) u_y = -\left(1 + \frac{u}{\sqrt{1 + u^2}}\right) \frac{x}{1 - x^2}.$$

Since this partial derivative is bounded on each x-interval $[0, b]$ where $0 < b < 1$, it follows from a standard theorem (e.g., [3; p. 550]) that the limiting equation has a unique solution $y = y(x)$ on [0,1) with prescribed $y(0) = y_0$. We turn now to the solution of this equation.

When $u = \frac{xy}{1-x^2}$, the quadratic equation for y' is

$$(y')^2 + \frac{2xy}{1 - x^2} y' - 1 = 0, \quad (0 \le x < 1). \tag{8}$$

With the substitutions $s = x^2$ and $p = -\frac{y'}{x}(> 0)$, equation (8) can be written

$$\frac{2y}{1-s} = p - \frac{1}{sp}, \quad \textit{where} \;\; p = -2\frac{dy}{ds}. \tag{9}$$

By differentiating with respect to s and eliminating y and $\frac{dy}{ds}$, we get the first-order equation

$$\frac{dp}{ds} = \frac{p}{s(s-1)(sp^2+1)} \quad (0 < s < 1) \tag{10}$$

which, although not standard, admits integration.

Indeed, with the *successive* substitutions $\frac{1}{s} = 1 + pq$, $p = v + q$, and $q = exp(w + \frac{v^2}{2})$, it reduces to the separable equation

$$\frac{dw}{dv} = e^w e^{\frac{v^2}{2}}.$$

This leads to an implicit solution in the form

$$(1-s)\int_v^c e^{\frac{t^2}{2}}\,dt = spe^{\frac{v^2}{2}} \quad (\textit{for appropriate } c) \tag{11}$$

where

$$v = p + \frac{s-1}{sp} \; (= 2y + sp). \tag{12}$$

[In principle, equations (11) and (12) determine p in terms of $s = x^2$ so that v and hence $y = \frac{1}{2}(v - sp)$ can be obtained as functions of x.]

We can derive qualitative information about our implicitly determined solution. First, note that the integration constant c is given by

$$c = v(0) = 2y(0) = 2y_0,$$

since as $s \searrow 0$, $sp = -xy' \to 0$. Moreover, for $s < 1$, we have $p(s) > 0$ and $\frac{dp}{ds} < 0$ by (10), so that as $s \nearrow 1$, $p(s)$ decreases to a limit $p_1 \geq 0$. In fact, $p_1 = 0$ since otherwise $v = p + \frac{s-1}{sp}$ has the positive limit $v_1 = p_1$ which violates our integral relation (11). It follows that y' is negative and approaches zero as $x \nearrow 1$ while $y(x)$ decreases to a *finite* limit y_1, say. (y_1 is negative, since $\frac{2y}{1-s} = p - \frac{1}{sp} \to -\infty$ as $s \nearrow 1$.)

Proposition 1. *Each solution curve $y = y(x)$ has a unique inflection point, and that point lies on the graph of the equation*

$$y = x\sqrt{\frac{1-x^2}{1+x^2}} \qquad (0 \leq x \leq 1). \tag{13}$$

PROOF: Observe that $y'' = 2\sqrt{s}\frac{d}{ds}\left(-\sqrt{s}p\right)$ so that, for $0 < s < 1$,

$$\begin{aligned} sgn\ y'' &= -sgn\left(\sqrt{s}\frac{dp}{ds} + \frac{1}{2\sqrt{s}}p\right) = -sgn\left(\frac{1}{s-1} + \frac{sp^2+1}{2}\right) \\ &= -sgn\left(\frac{1}{s-1} + 1 + \frac{ysp}{1-s}\right) = sgn\,(1-yp)\,, \end{aligned}$$

where we have used (9) and (10), together with the positivity of $p,\ s,\ sp^2+1$, and $1-s$.

We see that inflection occurs when $p = \frac{1}{y}$ or when $\frac{2y}{1-s} = \frac{1}{y} - \frac{y}{s}$ so that

$$y^2 = \frac{s(1-s)}{1+s} = x^2(\frac{1-x^2}{1+x^2})$$

as claimed. (Inflection must occur because near $s = 0 : 1 - yp < 0$ which cannot hold when y becomes negative, since $p > 0$.) □

The value x_0 where $y(x_0) = 0$ is of practical interest because it locates the boundary of the physical mirror. Conversely, it is clearly desirable to have x_0 as near 1 as possible and to know how large we must take $y_0 = y(0)$ to achieve this. However, when $x = x_0$, we see that $p = \frac{1}{x_0^2 p}$ and $v = x_0^2 p = x_0$. Then from our integral solution (with $c = 2y_0$) we get the transcendental relation

$$(1-x_0^2)\int_{x_0}^{2y_0} e^{\frac{t^2}{2}}\,dt = x_0 e^{\frac{x_0^2}{2}} \tag{14}$$

which implies that $y_0 \to +\infty$ as $x_0 \nearrow 1$.
If the integral in (14) is evaluated numerically, we find, for example, that when $x_0 = 0.999$, then $2.0030 < y_0 < 2.0031$.

Equation (13) for the locus of inflection points can be obtained directly. If we differentiate (8) with respect to x, set $y'' = 0$ and solve for y', we get

$$y' = \frac{-x}{y}.$$

Upon substituting this in (8), we recover (13). This approach also leads to an interesting geometrical fact. Consider the isocline associated with slope $m < -1$ obtained by replacing y' with m in (8). We can put the resulting equation in the form:

$$x\left(x + \left(\frac{2m}{1-m^2}\right)y\right) = 1$$

and we see that the isocline is a hyperbola having as asymptotes the y-axis and the line $y = \left(\frac{m^2-1}{2m}\right)x$. Moreover, the vertex of the relevant branch of the hyperbola has coordinates

$$x = \sqrt{\frac{m^2-1}{m^2+1}}, \quad y = \frac{-1}{m}\sqrt{\frac{m^2-1}{m^2+1}}.$$

But these coordinates satisfy (13), which characterizes an inflection point. Thus the locus of inflection points is the locus of the relevant vertices of the associated isoclines. In Figure 3 we exhibit the graphs of typical solutions and the locus of inflection points.

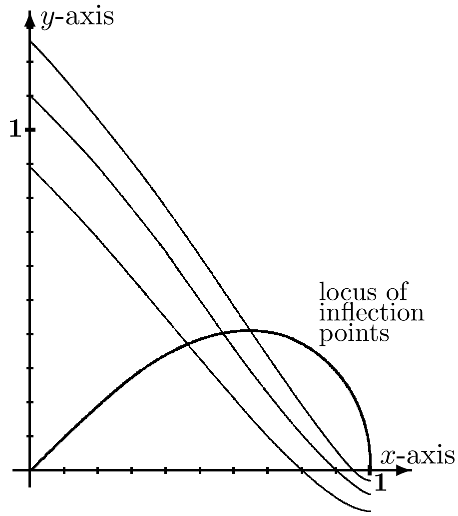

Figure 3

3. Solutions of the General Equation

For finite $Y > 0$, our differential equation (6) and (7) is considerably more complicated. However, it is straightforward to verify that $y = Y(1-x)$ gives the only decreasing linear solution. Now, $u = \frac{P}{Q}$, where $P = x[(Y-y)(yY-1)+x^2Y]$ and $Q = (Y-y)^2(1-x^2)+x^2y^2$ which is positive, if $0 < x < 1$ and $y < Y$. Consequently, for fixed $Y > 1$, $u(x,y)$ is bounded on each set $\{(x,y) : 0 \le x \le 1-\delta,\ y \le Y-\delta\}$ where $0 < \delta < 1$, as is the partial derivative

$$\frac{\partial u}{\partial y} = u_y = \frac{P_y}{Q} - u\frac{Q_y}{Q}.$$

From the argument used at the beginning of Section 2, we see that, for each $y_0 < Y$, there is a unique decreasing solution $y = y(x)$ of our equation on $[0,1)$ with the initial value $y(0) = y_0$. Moreover, the associated solution curves for distinct y_0 cannot intersect nor can they meet the open segment L between the points $(0,Y)$ and $(1,0)$ since its defining function, $y = Y(1-x)$, is also a solution of the equation. It follows that the solution must vanish at some $x_0 \in (0,1]$;

and conversely, for every $x_0 \in (0,1)$, there is a unique solution $y = y(x)$ on $[0,1)$ with $y(x_0) = 0$ and $y(0) \in (0, Y]$. In particular, we can take x_0 as near 1 as we please.

At an $x_0 \in (0,1)$, we have, from (6), that $u = -\frac{x_0}{Y}$ and, from (7), that $y'(x_0) = -\left(\sqrt{(\frac{x_0}{Y})^2 + 1} - \frac{x_0}{Y}\right) > -1$. But if $x_0 = 1$, the situation is less clear. In fact, when $Y > 1$, we note (see Figure 4) that the point $(1,0)$ ends the hyperbolic arc H defined by $(Y-y)(y-\frac{1}{Y}) + x^2 = 0$ $(0 \le x < 1,\ 0 < y \le \frac{1}{Y})$ along which, by (6) and (7), $u = 0$ and $y' = -1$. On the other hand, it also ends the linear solution segment L. Since no other solution segment is admissible, we see geometrically that, when $y_0 \in (\frac{1}{Y}, 1]$, the solution either crosses H with an intervening inflection point or it avoids H and L by having another inflection point. For $y_0 \in (1, Y)$, the solution curve must cross the circular arc C, defined by $x^2 + y^2 = 1$, $(0 \le x < x_L,\ y_L < y \le 1)$, where $y_L = -Y(x_L - 1)$, as shown. At the crossing point, (x_c, y_c), say, it can be easily verified

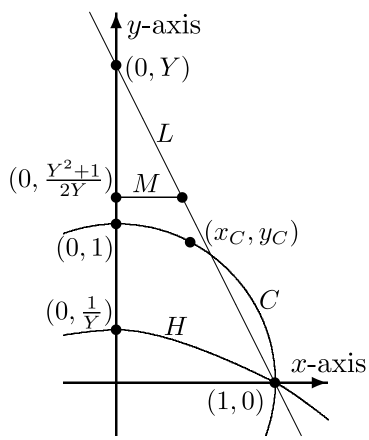

Figure 4

from (6) and (7) that the solution curve has slope $\frac{-y_c}{1-x_c} < -1$. Again, the curve either crosses H with slope -1 and thus has an inflection point, or it avoids H and L by tending (nonlinearly) toward (1,0) with an intervening inflection point. These arguments can be reinforced analytically, and they help establish our principal result:

Proposition 2. *Suppose $Y > 1$. Then, if $y_0 \in (\frac{1}{Y}, Y)$, the solution curve has a unique inflection point, and, if $y_0 \in (0, \frac{1}{Y}]$, the solution curve does not have an inflection point.*

(Of course, when $y_0 = Y$ the solution segment L has no inflection point.)

We only outline the arguments supporting the remaining assertions in this proposition. Note that along a solution curve $y(x)$ of (7) we have

$$y'' = -(1 + u(1+u^2)^{-\frac{1}{2}})u' = y'u'(1+u^2)^{-\frac{1}{2}}$$

where $u'(x) = \frac{d}{dx}u(x, y(x))$, so that $u' = u_x + u_y y'$. Hence in general, $sgn\, y'' = -sgn\, u'$, and at an inflection point, $u' = 0$ with $u_x u_y \geq 0$ (since $y' < 0$). Now, when (6) is used for fixed Y, then formally

$$u' = R(x, y, y')$$

where R is a rational function of its variables that is linear in $y' = -u - \sqrt{1+u^2}$. By direct computation, we can show that $u = xY$ and $u'(x) \neq 0$ at points on the horizontal open segment M of height $m = \frac{Y^2+1}{2Y}$ between L and the y-axis. Moreover, since $u(0) = 0$, it is easy to verify that $sgn\; y''(0) = sgn(\frac{1}{Y} - y_0)$ when $y_0 < Y$. If we further differentiate and set $y'' = u' = 0$, we find (eventually) that, with P and Q as before,

$$sgn\, y'''(x) = sgn\,\{(y-m)[2x(Y - y - x^2Y) + P - \sqrt{P^2+Q^2}]\},$$

where, for $0 < x < 1 < Y$, the second factor is not positive and it is strictly negative unless $y = Y(1 \pm x)$. When $y_0 \in (0, \frac{1}{Y})$, $y''(0) > 0$ and it follows that y'' cannot vanish at a 'first' x value since there $y'''(x) > 0$; the associated solution curves have no inflection points. We can extend this argument to the case $y_0 = \frac{1}{Y}$ where $y''(0) = 0$ but $y'''(0) > 0$ since then $y''(x) > 0$, for $0 < x \leq x_1$, with $y(x_1) < \frac{1}{Y}$.

When $y_0 \in (\frac{1}{Y}, m]$, y''' will be positive at every inflection point, so that there cannot be more than one. Finally, if $y_0 \in (m, Y)$, then $y_0 > m$ and $y''(0) < 0$; hence, y'' cannot vanish at a 'first' x with $y(x) > m$ since there $y'''(x) < 0$. It follows that all inflection points must occur below M, and again we conclude that there is at most one.□

By straightforward extension of these arguments using L'Hospital's rule as needed, we can also prove:

Corollary 1. *L is the only solution curve that either originates at $(0, Y)$ or terminates at $(1, 0)$.*

In particular, there cannot be a "perfect" mirror that inverts the entire unit disk. However, for specific Y, we can use standard methods to obtain numerical solutions to our equations; and in Figure 5 we present representative solution curves when $Y = 10$, for values of $x_0 = 0.8$, 0.9, 0.95 with corresponding values of $y_0 = 0.887$, 1.088, 1.245. In particular, the numerical solution with $x_0 = 0.95$ (so $y_0 = 1.245$) gives the profile of a mirror that should faithfully invert the region exterior to the disk of 5 inch diameter when viewed from a height of about 2 feet.

It seems feasible to manufacture such a mirror on a computer-directed lathe[1].

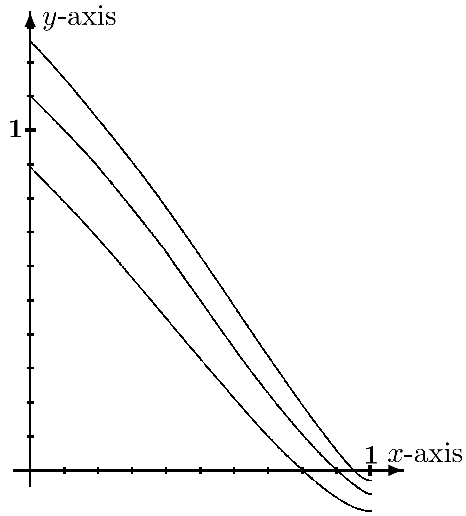

Figure 5

References

1. Davis, P. J. *The Thread: A Mathematical Yarn.* The Harvester Press, Birkhäuser, Boston, 1983.

2. Kempe, A. B. *How to Draw a Straight Line.* National Council of Teachers of Mathematics, Reston, VA, 1977.

3. Simmons, G. F. *Differential Equations with Applications and Historical Notes, Second Edition.* McGraw-Hill, New York, 1991.

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