Neohookean Deformations of Annuli,

Existence, Uniqueness and Radial Symmetry

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Abstract

Let $\mathbb{X} = \{x \in \mathbb{R}^2\,;\;\; r < |x| < R\}$ and $\mathbb{Y} = \{y \in \mathbb{R}^2\,;\;\; r_* < |y| < R_*\}$ be annuli in the plane. We consider the class $\mathscr{F}(\mathbb{X}, \mathbb{Y})$ of all orientation preserving homeomorphisms $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in the Sobolev space $\mathscr{W}^{1,2}(\mathbb{X}, \mathbb{Y})$ which keep the boundary circles in the same order. This means that $\lim\limits_{|x|\searrow r} |h(x)| = r_*$ and $\lim\limits_{|x|\nearrow R} |h(x)| = R_*$. We study the neohookean energy integral

$$\mathcal{E}[h] = \int_{\mathbb{X}} |Dh|^2 + \Phi(\det Dh) \qquad \text{for} \;\; h \in \mathscr{F}(\mathbb{X}, \mathbb{Y}) \tag{1}$$

where $\Phi \in \mathscr{C}^\infty(0, \infty)$ is positive and strictly convex. We assume in addition that the function $\Psi(z) = 1/\ddot{\Phi}(z)$ and its derivative extend continuously to $[0, \infty)$, with $\Psi(0) = 0$. Then we prove:

THEOREM 1. *The minimum of energy within the class $\mathscr{F}(\mathbb{X}, \mathbb{Y})$ is attained for a radial map $h(x) = H(|x|)\frac{x}{|x|}$. The minimizer is $\mathscr{C}^\infty$-smooth and is unique up to a rotation of the annuli.*

We believe that not only the result but also some novelties in the computation might gain a particular interest.

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1 Introduction

In this note we study a concrete extremal problem motivated by recent remarkable relations between Geometric Function Theory [20, 19], mappings of finite distortion, and the theory of nonlinear elasticity [1, 4, 12], hyperelastic deformations in particular. Both theories are governed by variational principles. Here we confine ourselves to deformations of bounded planar domains, denoted by $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^2$. Let a homogeneous isotropic elastic body in its reference configuration $\mathbb{X}$ occupy the deformed configuration $\mathbb{Y}$. The general law of hyperelasticity tells us that there exists a stored energy function $E : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ that characterizes the elastic and mechanical properties of the material. The subject of the study are orientation preserving homeomorphisms $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of smallest energy;

$$\mathcal{E}[h] = \int_{\mathbb{X}} E\left(|Dh|^2, \det Dh\right) \,\mathrm{d}x \tag{2}$$

called *extremal deformations.* Let us assume, as in many practical situation, that the stored energy function $E : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is convex and increasing in the first variable. The elasticity theory deals with deformations with prescribed values on $\partial\mathbb{X}$. In Geometric Function Theory, on the other hand , we do not specify the boundary values of h. For example extremal Teichmüller quasiconformal mappings are not constrained on the boundary; the bounday does not even exist for compact Riemann surfaces. Nevertheless, such mappings can be realized physically as deformations of a material confined in a box, allowing tangential slipping along the boundary, [7]. A natural question is whether or not $\mathcal{E}[h]$ is finite and, if so, whether or not it assumes its minimum value among all homeomorphism $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in a given homotopy class [21]. In dimension $n = 2$ these are the issues of the theory of harmonic maps between Riemann surfaces, [14, 11, 31, 25, 26, 27, 28, 29, 33, 34, 39]. These are minimizers of the Dirichlet integral, also called conformal energy

$$\mathcal{E}_D[h] = \int_{\mathbb{X}} |Dh(x)|^2 \,\mathrm{d}x \tag{3}$$

If $\mathbb{X}$ and $\mathbb{Y}$ are simply connected, the Riemann mapping theorem asserts that there exist a conformal deformation $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$. This map is certainly a minimizer of the Dirichlet integral. Indeed, for every h we have

$$\mathcal{E}_D[h] = \int_{\mathbb{X}} |Dh|^2 \;\geqslant\; 2\int_{\mathbb{X}} J(x,h)\,\mathrm{d}x \;=\; 2\,|\mathbb{Y}|\,, \qquad J(x,h) = \det Dh \tag{4}$$

with equality if and only if h is conformal. The differential form $J(x,h)\,\mathrm{d}x$, will be recognized as being a free Lagrangian when dealing with more general energy integrals. Let us now look at the example in which the existence of deformations of finite conformal energy is lacking [22]. The first nontrivial situation that two topologically equivalent domains are not of the same conformal type is that of concentric annuli

$$\mathbb{X} = \{x \in \mathbb{R}^2\,; \quad r < |x| < R\} \qquad \text{and} \qquad \mathbb{Y} = \{y \in \mathbb{R}^2\,; \quad r_* < |y| < R_*\} \quad (5)$$

These annuli are of different conformal type unless the ratio of two radii is the same for the both annuli (Schottky's theorem). Now suppose that the reference annulus $\mathbb{X}$ is substantially fatter than $\mathbb{Y}$, precisely

$$\frac{R}{r} > \frac{R_*}{r_*} + \sqrt{\frac{R_*^2}{r_*^2} - 1}\,, \qquad \text{or equivalently} \quad \frac{R_*}{r_*} < \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right) \qquad (6)$$

Although it is not obvious at all, the minimizer within all weak $\mathscr{W}^{1,2}$-limits of homeomorphims $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ takes the form:

$$h(z) = \begin{cases} r_* \frac{z}{|z|}\,, & r < |z| < \rho \quad \text{-not harmonic, squeezing map} \\ r_* \left(\frac{z}{2\,\rho} + \frac{\rho}{2\,\overline{z}}\right)\,, & \rho < |z| < R \quad \text{-critical harmonic Nitsche map} \end{cases}$$

Here ρ is determined by the so-called critical Nitsche equation

$$\frac{R_*}{r_*} = \frac{1}{2}\left(\frac{R}{\rho} + \frac{\rho}{R}\right) \qquad (7)$$

Note that $J(z,h) \equiv 0$ for $r \leqslant |z| \leqslant \rho$. This minimizer is actually unique up to the rotation of annuli, [22]. In light of this result it seems plausible that under the relation (6) there is no univalent harmonic map from $\mathbb{X}$ onto $\mathbb{Y}$. (Nitsche Conjecture [31], 1962). The reader may wish to observe that in the critical case; that is, for $r = \rho$, the Jacobian determinant vanishes when we approach the inner circle of the annulus $\mathbb{X}$. That is why, beyond this circle the harmonic extension of h is no longer injective. The situation is dramatically different if the mappings in question have integrable distortion function [24, 2, 17, 18]; that is,

$$K(x,h) \stackrel{\text{def}}{=\!=} \frac{|Dh(x)|^2}{J(x,h)} \in \mathscr{L}^1(\mathbb{X}) \qquad (8)$$

This brought us to a study of the variational integral, called total mean distortion, such as

$$\mathcal{E}[h] = \int_{\mathbb{X}} \left[|Dh(x)|^2 \,+\, K(x,h)\right]\,\mathrm{d}x \qquad (9)$$

Roughly speaking, $\mathscr{L}^1$-integrability of $K(x,h)$ prevents a minimizer from having vanishing Jacobian. Indeed, we have shown in [23] that

THEOREM 2. *Among all $\mathscr{W}^{1,2}$-limits of homeomorphims $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ the smallest total mean distortion is attained on a $\mathscr{C}^\infty$-diffeomorphism. This radial map is a unique minimizer up to conformal automorphisms of the annuli.*

This result is a step toward a very coherent development of Teichmüller theory of deformations with unbounded distortion. We refer the interested reader to [23]. Numerous isotropic polyconvex stored energies were introduced and employed effectively in nonlinear elasticity [4, 5, 6, 8, 9, 10, 13, 15, 30, 32, 35, 36, 37, 38, 40]. For instance, adding a negative power of the Jacobian determinant to the Dirichlet integrand will result in the injectivity of the minimizers, at least for the extremal mappings between annuli. To this category of right variational problems we include the energy integrals of the form

$$\mathcal{E}[h] = \int_{\mathbb{X}} \big[|Dh(x)|^2 \, + \, \Phi\big(J(x,h)\big) \big] \, \mathrm{d}x \tag{10}$$

where the additional term, referred to as the *Jacobian ingredient*, is convex. Let us hope for the injectivity of deformations with smallest energy. To this effect we try to prevent the Jacobian from having arbitrarily small values in $\mathbb{X}$. This can be achieved by assuming, in addition to convexity, that the function $z \to \Phi(z)$ is decreasing; the smaller Jacobian the larger ingredient to the total energy. There are examples abound in the polyconvex calculus of variations;

$$\mathcal{E}[h] = \int_{\mathbb{X}} \big[|Df(x)|^2 \, + \, J(x,h)^p \big] \, \mathrm{d}x \, , \qquad p < 0 \tag{11}$$

$$\mathcal{E}[h] = \int_{\mathbb{X}} \big[|Df(x)|^2 \, - \, J(x,h)^q + 1 \big] \, \mathrm{d}x \, , \qquad 0 < q < 1 \tag{12}$$

The examples of the first kind are usually referred to as neohookean integrals, because the integrand blows up as $J(x,h)$ approaches zero [4, 8, 9, 32, 40]. More precisely, what is really needed is the following condition concerning the second derivative of Φ,

$$\lim_{z \searrow 0} \ddot{\Phi}(z) = \infty \tag{13}$$

However to ease rather involved computation we actually assume that

$$\Phi \in \mathscr{C}^\infty(0,\infty) \, , \qquad \ddot{\Phi} > 0 \quad \text{and} \quad \lim_{z \to 0} \ddot{\Phi}(z) = \infty \tag{14}$$

Furthermore, we assume:

$$\text{the function} \quad 1/\ddot{\Phi} \quad \text{and its derivative extend continuously to} \quad [0,\infty) \tag{15}$$

Remark 1. The latter condition is motivated by our considerations in Section 3, where we establish $\mathscr{C}^1$-dependence in a parameter for the solutions of a differential equation. It seems that the condition (15) is redundant, but we shall not take trouble with more involved arguments. We have made special effort to present the sometimes unpredictable computational details as simply and clear as possible, see the appendix.

The fundamental concept in our approach is that of *free Lagrangians* which narrows a bit the familiar notion of null Lagrangians, [4]. A free Lagrangian is a nonlinear differential form $L(x,h,Dh)\,\mathrm{d}x$ defined on homeomorphisms $h: \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ whose integral depends only on the homotopy class of h, regardless of its boundary values, [22, 23, 3]. The key to the proof of Theorem 1 is a sharp pointwise estimate of the stored-energy function by means of free Lagrangians, say

$$|Dh(x)|^2 + \Phi\big(J(x,h)\big) \geqslant L(x,h,Dh) \tag{16}$$

for every admissible homeomorphism $h: \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$. Of course one should first predict the extremal map and then ensure that for this map equality takes place in (16). Finding such a free Lagrangian $L(x,h,Dh)$ is truly a work of art.

At first sight Theorem 1 appears to pose no challenge. However, the seemingly natural attempt via symmetrization fails. Upon natural geometric speculations one would think that the rotational symmetry of the annuli and the rotational invariance of the stored-energy function must yield radial symmetry of the extremal deformations. This is not always the case, and we have some surprise for the reader [22].

THEOREM 3. *For each $n \geqslant 4$ there are annuli $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ such that no radial mapping minimizes the n-harmonic energy*

$$\mathcal{E}[h] = \int_{\mathbb{X}} |Dh(x)|^n \,\mathrm{d}x \tag{17}$$

2 Free Lagrangians and the Idea of the Proof

We shall work with one particular homotopy class $\mathscr{F}(\mathbb{X}, \mathbb{Y})$ of homeomorphisms $h: \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ between annuli, one defined in Abstract. In this context a *free Lagrangian* refers to a differential 2-form $L(x,h,Dh)\,\mathrm{d}x$, formulated for $h \in \mathscr{F}(\mathbb{X}, \mathbb{Y})$, whose integral over $\mathbb{X}$ does not depend on a particular choice of the mapping $h \in \mathscr{F}(\mathbb{X}, \mathbb{Y})$. Naturally, polar coordinates

$$x = te^{i\theta}\,, \qquad r < t < R \quad \text{and} \quad 0 \leqslant \theta < 2\pi \tag{18}$$

are best suited for dealing with mappings of planar annuli. The radial (normal) and angular (tangential) derivatives of $h : \mathbb{X} \to \mathbb{Y}$ are defined by

$$h_N(x) = \frac{\partial h(te^{i\theta})}{\partial t}\,, \qquad t = |x| \tag{19}$$

and

$$h_T(x) = \frac{1}{t}\frac{\partial h(te^{i\theta})}{\partial \theta}\,, \qquad t = |x| \tag{20}$$

For a general map $h \in \mathscr{F}(\mathbb{X}, \mathbb{Y})$ we have the formulas

$$|Dh|^2 = |h_N|^2 + |h_T|^2 \qquad \text{and} \qquad J(x,h) = \operatorname{Im}\left(h_T \overline{h_N}\right) \leqslant |h_T|\,|h_N| \tag{21}$$

We shall make use of four free Lagrangians.

- A function in x;

$$L(x,h,Dh)\,\mathrm{d}x = M(x)\,\mathrm{d}x \tag{22}$$

  Thus, for all $h \in \mathscr{F}(\mathbb{X}, \mathbb{Y})$ we have

$$\int_{\mathbb{X}} L(x,h,Dh)\,\mathrm{d}x = \int_{\mathbb{X}} M(x)\,\mathrm{d}x \tag{23}$$

- Pullback of a form in $\mathbb{Y}$ via a given mapping $h \in \mathscr{F}(\mathbb{X}, \mathbb{Y})$;

$$L(x,h,Dh)\,\mathrm{d}x = N(|h|)\,J(x,h)\,\mathrm{d}x\,, \qquad \text{where } N \in \mathscr{L}^1(r_*, R_*) \tag{24}$$

  Thus, for all $h \in \mathscr{F}(\mathbb{X}, \mathbb{Y})$ we have

$$\int_{\mathbb{X}} L(x,h,Dh)\,\mathrm{d}x = \int_{\mathbb{Y}} N(|y|)\,\mathrm{d}y = 2\pi \int_{r_*}^{R_*} N(\tau)\,\tau\,\mathrm{d}\tau \tag{25}$$

- A radial free Lagrangian

$$L(x,h,Dh)\,\mathrm{d}x = A\big(|h|\big)\frac{|h|_N}{|x|}\,\mathrm{d}x\,, \qquad \text{where } A \in \mathscr{L}^1(r_*, R_*) \tag{26}$$

  Thus, for all $h \in \mathscr{F}(\mathbb{X}, \mathbb{Y})$ we have

$$\int_{\mathbb{X}} L(x,h,Dh)\,\mathrm{d}x = 2\pi \int_r^R A\big(|h|\big)\frac{\partial |h|}{\partial \rho}\,\mathrm{d}\rho = 2\pi \int_{r_*}^{R_*} A(\tau)\,\mathrm{d}\tau \tag{27}$$

- An angular free Lagrangian

$$L(x,h,Dh) = B\big(|x|\big)\operatorname{Im}\frac{h_T}{h}\,, \qquad \text{where } B \in \mathscr{L}^1(r,R) \tag{28}$$

  Thus, for all $h \in \mathscr{F}(\mathbb{X}, \mathbb{Y})$ we have

$$\int_{\mathbb{X}} L(x,h,Dh)\,\mathrm{d}x = \int_r^R \frac{B(t)}{t}\left(\int_{|x|=t} \frac{\partial \operatorname{Arg} h}{\partial \theta}\,\mathrm{d}\theta\right)\mathrm{d}t = 2\pi \int_r^R \frac{B(t)}{t}\,\mathrm{d}t \tag{29}$$

The idea behind our use of these free Lagrangians is to establish a general subgradient type inequality for the integrand with two independent parameters $t \in (r, R)$ and $\tau \in (r_*, R_*)$

$$|Dh|^2 + \Phi\,(\det Dh) \quad \geqslant \quad \alpha(\tau)\,\frac{|h|_N}{t} + \beta(t)\frac{|h|}{\tau}\,\mathrm{Im}\,\frac{h_T}{h} + \gamma(\tau)\det Dh + \delta(t)$$

where the coefficients α and γ are functions in the variable $r_* < \tau < R_*$, while β and δ are functions in the variable $r < t < R$. We will substitute $t = |x|$ and $\tau = |h(x)|$ to obtain free Lagrangians in the right hand side. Some specific relations between these coefficients are needed in order to have equality for a radial equilibrium solution. Finding such coefficients requires deep geometric insight into this problem. However, we do not present here the details behind our explicit formulas for α, β, γ and δ. The reader will have to take those formulas for granted.

3 Radial Equilibrium Solutions

We begin with the discussion of stationary points of the energy

$$\mathcal{E}[h] = \int_{\mathbb{X}} |Dh|^2 + \Phi\,(\det Dh) \tag{30}$$

among radial mappings $h(x) = H(|x|)\frac{x}{|x|}$ in the class $\mathscr{F}(\mathbb{X}, \mathbb{Y})$. We find that

$$|Dh|^2 = [\dot{H}(t)]^2 + \frac{[H(t)]^2}{t^2}\,, \qquad\qquad t = |x| \tag{31}$$

and

$$\det Dh = \frac{\dot{H}(t)\,H(t)}{t}\,, \qquad\qquad t = |x| \tag{32}$$

Formal computation reveals that the radial equilibrium equation for $H = H(t)$, $r < t < R$ must take the form

$$\ddot{H} = \Big(H - t\dot{H}\Big)\;\frac{H\,\dot{H}\,\ddot{\Phi}\; + \;2\,t}{t\,H^2\,\ddot{\Phi}\; + \;2\,t^3} \tag{33}$$

It is therefore of interest to look at the solutions to this equation. We study this equation and the solutions without proclaiming their relation with the mappings of smallest energy. We though continue to refer to them as radial equilibria.

3.1 Construction of a radial equilibrium

We are interested in the $\mathscr{C}^2$-solutions to (33) with $\dot{H}(t) > 0$, which satisfy the boundary constraints $H(r) = r_*$ and $H(R) = R_*$. For the subsequent considerations we will need only know one such solution.

Proposition 1. *Let $\Phi : (0,\infty) \to (0,\infty)$ be $\mathscr{C}^\infty$-smooth and convex. Assume, in addition, that the function $\Psi(z) = 1/\ddot{\Phi}(z)$ and its derivative extend continuously to $[0,\infty)$, with $\Psi(0) = 0$. Then the boundary value problem*

$$\begin{cases} \ddot{H} = (H - t\dot{H})\left(H\dot{H}\ddot{\Phi} + 2t\right)\left(tH^2\ddot{\Phi} + 2t^3\right)^{-1} \quad , \quad \ddot{\Phi} = \ddot{\Phi}\left(\frac{H\dot{H}}{t}\right) \\ H(r) = r_*\,, \qquad H(R) = R_* \end{cases} \tag{34}$$

admits a solution $H \in C^\infty(a,\infty)$ with $\dot{H} > 0$ in an interval $(a,\infty) \supset [r,\infty)$.

Theorem 1, to be established later, combines with this proposition to show that in fact any solution of (34) in $\mathscr{W}^{1,2}(r,R)$ is unique and, therefore, coincides with H.

Proof of Proposition 1. First, we consider so-called auxiliary equation

$$\dot{V}(s) = \frac{2V \,-\, 2s^2}{s^3\,\ddot{\Phi}(V) \,+\, 2s} \overset{\text{def}}{=\!=} G(s,V)\,, \qquad s > 0 \tag{35}$$

for a positive function $V \in \mathscr{C}^\infty(0,\infty)$. Then we will look for H by solving the first order equation

$$\begin{cases} \dot{H} = \frac{t}{H}\, V\left(\frac{H}{t}\right)\,, \qquad t > a \\ H(r) = r_* \end{cases} \tag{36}$$

With a suitable choice of the solution $V = V(s)$ we will be able to ensure that $H(R) = R_*$. It is easy to verify, via differentiation of (36), that H satisfies the equation (33). Let us first extend the equation (35) to allow for negative solutions $V : (0,\infty) \to \mathbb{R}$. Precisely, we set

$$\dot{V}(s) = G(s,V)\,, \qquad \text{for} \quad V : \mathbb{R}_+ \to \mathbb{R} \tag{37}$$

where $G(s,-v) = -G(s,v)$ if v is negative. Note that $G : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is $\mathscr{C}^1$-smooth. It is exactly at this point that we appeal to the hypotheses on Φ in Proposition 1. Another important feature of $G(s,v)$ that we wish to observe is the linear growth in v,

$$|G(s,v)| \leqslant \frac{|v|}{s} + s \tag{38}$$

The reader may wish to consult the book by Hartman [16] for basic facts we need here. Recall the Picard-Lindelöf Theorem. Accordingly, through any point $(s_\circ, v_\circ) \in \mathbb{R}_+ \times \mathbb{R}$ there passes exactly one local $\mathscr{C}^2$-integral curve. Then the extension theorem tells us that such a local solution $V = V(s)$ extends uniquely (as a solution) to its maximal interval of existence. Denote this interval by (a,b), where $0 \leqslant a < b \leqslant \infty$. We aim to show that $a = 0$ and $b = \infty$. For this we recall that the maximal integral curve must approach the boundary of

$\mathbb{R}_+ \times \mathbb{R}$, including infinity. Suppose that, on the contrary, $a > 0$. Then $V(s)$ must be unbounded near a. However, the differential inequality

$$\left|\dot{V}(s)\right| \leqslant \frac{|V(s)|}{s} + s \tag{39}$$

precludes the existence of unbounded solutions near any point $a > 0$. Thus, $a = 0$. Similarly we conclude that $b = \infty$. Next we parametrize all solutions by the initial datum

$$V(1) = \lambda\,, \qquad \text{where} \;\; \lambda \in \mathbb{R} \tag{40}$$

Let us denote such solution by $V_\lambda = V_\lambda(s)$. The function $(s, \lambda) \to V_\lambda(s)$ is at least of class $\mathscr{C}^1(\mathbb{R}_+ \times \mathbb{R})$, see [16, Ch. V, Corollary 4.1] and $V_0(s) \equiv 0$. This particular integral curve separates the upper half $\mathbb{R}_+ \times \mathbb{R}_+$ from the lower half $\mathbb{R}_+ \times \mathbb{R}_-$ of the region $\mathbb{R}_+ \times \mathbb{R}$. Thus the integral curves corresponding to $\lambda > 0$ lie in the upper half, as they cannot cross the integral curve $V_0(s) \equiv 0$, see Figure 1.

Figure 1: The auxiliary solutions.

It is important to notice that for every fixed $s > 0$ the function $\lambda \to V_\lambda(s)$ continuously increases from $-\infty$ to $+\infty$.

From now on we will confine ourselves to integral curves in the first quadrant; that is, with $\lambda > 0$. Since $G \in \mathscr{C}^\infty(\mathbb{R}_+, \mathbb{R}_+)$ the function $(s, \lambda) \to V_\lambda(s)$ is $\mathscr{C}^\infty$-smooth in $(\mathbb{R}_+, \mathbb{R}_+)$, see again [16, Ch. V, Corollary 4.1]. Having disposed of the functions V_λ, we return to the equation for H,

$$\begin{cases} \dot{H}(t) = \frac{t}{H}\, V_\lambda\left(\frac{H}{t}\right) \\ H(r) = r_* \end{cases} \tag{41}$$

The existence and uniqueness of the solution and its maximal interval are established by the Picard-Lidelöf theorem. Here we note that for λ fixed $V_\lambda(s)$ is bounded so both $\dot{H}$ and H are locally bounded. Therefore, the maximal integral curve $(\tau = H(t))$ hits the boundary of $\mathbb{R}_+ \times \mathbb{R}_+$. Since H is increasing the left endpoint of this curve is either $(0, y)$ with $0 \leqslant y < r_*$ or $(x, 0)$ with $0 \leqslant x < r$. In any case the maximal interval of existence of $H = H(t)$, denoted by (a_λ, ∞), contains $[r, \infty)$. We denote by $H_\lambda = H_\lambda(t)$ this maximal solution, see Figure 2.

We also have smooth dependence on the parameter λ; so the function $\lambda \to H_\lambda(R)$ is continuous. Note that $\lim\limits_{\lambda \to 0} H_\lambda(R) = r_* < R_*$, because $H_\lambda(r) = r_*$ and $\dot{H}_\lambda(t)$ goes to zero uniformly for $r \leqslant t \leqslant R$ as $\lambda \to 0$. Now, in order to find λ

Figure 2: The radial equilibria.

for which $H_\lambda(R) = R_*$ it suffices to show that $H_\lambda(R)$ assumes arbitrarily large values. Suppose, to the contrary, that $H_\lambda(R) \leqslant M$ for all $0 \leqslant \lambda < \infty$, where M is a constant. We have,

$$H_\lambda(R) \;=\; H_\lambda(r) + \int_r^R \dot{H}_\lambda(t)\,\mathrm{d}t = r_* + \int_r^R \frac{t}{H_\lambda(t)} V_\lambda\left(\frac{H_\lambda(t)}{t}\right)\mathrm{d}t \quad (42)$$

Note that the values of $\frac{H_\lambda(t)}{t}$ lie in a specific interval $[\alpha, \beta]$ independent of λ. Precisely, we have $\alpha = \frac{r_*}{R} \leqslant \frac{H_\lambda(t)}{t} \leqslant \frac{H_\lambda(R)}{r} \leqslant \frac{M}{r} = \beta$. Thus for $\lambda > \left(\frac{M}{R}\right)^2$ we have

$$H_\lambda(R) \geqslant \frac{(R-r)\,r}{M} \inf_{\alpha \leqslant s \leqslant \beta} V_\lambda(s) \geqslant \frac{(R-r)\,r}{M} V_\lambda(\alpha) \quad (43)$$

Hence $\limsup_{\lambda\to\infty} H_\lambda(R) = \infty$, as required.

Let $\lambda > 0$ be one of the parameters for which $H_\lambda(R) = R_*$. Then $H_\lambda \in \mathscr{C}^\infty(a_\lambda, \infty)$. Note that we did not claim the uniqueness of such parameter λ, though it is most likely the case.

3.2 The elasticity of the radial stretching

The equation (33) can be written as

$$(H - t\dot{H})' = (H - t\dot{H})\, M(t)\,, \quad M(t) = -\frac{t\, H\, \dot{H}\, \ddot{\Phi} \;+\; 2t^2}{t\, H^2\, \ddot{\Phi} \;+\; 2t^3} \quad (44)$$

which we solve for $H - t\dot{H}$

$$H - t\dot{H} = C \exp\left(\int_r^t M(s)\,\mathrm{d}s\right), \qquad C \quad \text{- a constant} \quad (45)$$

In particular, $H - t\dot{H}$ cannot change sign. In this way we encounter three cases concerning the elasticity function $\eta(t) = \eta_{_H}(t) \overset{\text{def}}{=\!=} \frac{t\,\dot{H}}{H}$

$$\eta_{_H}(t) = \frac{t\dot{H}}{H} < 1\,, \quad \ddot{H} > 0 \qquad \text{for all } r < t < R \quad \textit{(inelastic case)} \quad (46)$$

$$\eta_{_H}(t) = \frac{t\,\dot{H}}{H} = 1\,, \quad \ddot{H} \equiv 0 \qquad \text{for all } r < t < R \quad \textit{(conformal case)} \quad (47)$$

$$\eta_{_H}(t) = \frac{t\,\dot{H}}{H} > 1\,, \quad \ddot{H} < 0 \qquad \text{for all } r < t < R \quad \textit{(elastic case)} \quad (48)$$

These cases can actually be identified directly by means of the conformal moduli of the annuli $\mathbb{X}$ and $\mathbb{Y}$. Indeed, we have the logarithmic average identity

$$\frac{\int_r^R \eta_{_H}(t)\,\frac{\mathrm{d}t}{t}}{\int_r^R \frac{\mathrm{d}t}{t}} = \frac{\log \frac{R_*}{r_*}}{\log \frac{R}{r}} \tag{49}$$

Hence

- $\eta_{_H}(t) < 1$ iff $\frac{R_*}{r_*} < \frac{R}{r}$ (the deformed configuration is conformally thinner than the reference annulus)
- $\eta_{_H}(t) = 1$ iff $\frac{R_*}{r_*} = \frac{R}{r}$ (the domains $\mathbb{X}$ and $\mathbb{Y}$ are conformally equivalent)
- $\eta_{_H}(t) > 1$ iff $\frac{R_*}{r_*} > \frac{R}{r}$ (the deformed configuration is conformally fatter than the reference annulus)

Before we proceed to the proof of Theorem 1 let us state the analogous equations for the inverse function of H. We reserve the notation $F = F(\tau)$, $r_* < \tau < R_*$, for $F = H^{-1}$. That is, $F\big(H(t)\big) \equiv t$ and $H\big(F(\tau)\big) \equiv \tau$. we have the following useful relations, where $\tau = H(t)$ and $t = F(\tau)$;

$$\dot{H}(t)\,\dot{F}(\tau) = 1\,, \qquad \dot{H}^2 + \frac{H^2}{t^2} = \frac{1}{\dot{F}^2} + \frac{\tau^2}{F^2}\,, \qquad \frac{\dot{H}H}{t} = \frac{\tau}{F\dot{F}}\,, \tag{50}$$

$$\eta_{_H}(t)\,\eta_{_F}(\tau) = 1\,, \qquad\qquad \eta_{_F}(\tau) \overset{\text{def}}{=\!=} \frac{\tau\dot{F}}{F} \tag{51}$$

Here and in the sequel H will always be a function in $t \in [r, R]$ and F be a function in $\tau \in [r_*, R_*]$. The above three cases translate into:

- inelastic case

$$\eta_{_F}(\tau) > 1\,, \quad \ddot{F}(\tau) < 0\,, \quad \frac{R_*}{r_*} < \frac{R}{r}\,, \quad F(\tau) < \tau\dot{F}(\tau)$$

- conformal case

$$\eta_{_F}(\tau) \equiv 1\,, \quad \ddot{F}(\tau) \equiv 0\,, \quad \frac{R_*}{r_*} = \frac{R}{r}\,, \quad F(\tau) = \tau\dot{F}(\tau)$$

- elastic case

$$\eta_{_F}(\tau) < 1\,, \quad \ddot{F}(\tau) > 0\,, \quad \frac{R_*}{r_*} > \frac{R}{r}\,, \quad F(\tau) > \tau\dot{F}(\tau)$$

The radial equilibrium equation (33) can be expressed in terms of F as

$$\ddot{F}(\tau) = \frac{(F - \tau\dot{F})\,\dot{F}}{F}\,\frac{2\,F\,\dot{F}\;+\;\tau\,\ddot{\Phi}}{\tau^2\,\ddot{\Phi}\;+\;2F^2}\,, \qquad \ddot{\Phi} = \ddot{\Phi}\left(\frac{\tau}{F\,\dot{F}}\right) \tag{52}$$

4 Proof of Theorem 1

The proof falls into three parts. Let us begin with the easy one.

4.1 Conformal case

The case $\frac{R_*}{r_*} = \frac{R}{r}$ is classic. To this effect we observe that $|Dh|^2 \geqslant 2 \det Dh$. Using Jensen's inequality see that

$$\mathcal{E}[h] \quad \geqslant \quad 2 \int_{\mathbb{X}} \det Dh \, + \, |\mathbb{X}| \, \Phi \left(\frac{1}{|\mathbb{X}|} \int_{\mathbb{X}} \det Dh \right) = 2|\mathbb{Y}| + |\mathbb{X}| \, \Phi \left(\frac{|\mathbb{Y}|}{|\mathbb{X}|} \right)$$

Equality occurs here if and only if $|Dh|^2 = 2 \det Dh$ and $\det Dh =$ *constant*. The first condition tells us that h must be conformal, while the second equation means that h has constant derivative. Therefore, using complex variables, the solution takes the form $h(z) = az$, where a is any complex number of modulus $|a| = \frac{r_*}{r} = \frac{R_*}{R}$.

4.2 Elastic case

Choose and fix a radial mapping $h_\circ = H\big(|x|\big) \frac{x}{|x|}$ in $\mathscr{F}(\mathbb{X}, \mathbb{Y})$, where H satisfies the equation (33). Suppose we are given an arbitrary mapping $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in $\mathscr{F}(\mathbb{X}, \mathbb{Y})$. We shall give a series of sharp estimates involving h and Dh, each of which becomes an equality if $h = h_\circ$. Let us introduce the following functions:

i)

$$a = a(\tau) = \frac{1}{\dot{F}^2(\tau)} + \frac{\tau^2}{F^2(\tau)}\,, \qquad r_* \leqslant \tau \leqslant R_*$$

Thus $a\big(|h(x)|\big)$ is a well defined function on $\mathbb{X}$, and it equals $|Dh(x)|^2$ if $h = h_\circ(x)$.

ii)

$$b = b(\tau) = \frac{\tau}{F(\tau)\, \dot{F}(\tau)}\,, \qquad r_* \leqslant \tau \leqslant R_*$$

Thus $b\big(|h(x)|\big) = \det Dh(x)$ when $h = h_\circ(x)$.

iii)

$$p = p(\tau) = \frac{\tau\, \dot{F}(\tau)}{F(\tau)} < 1\,, \qquad r_* \leqslant \tau \leqslant R_*$$

We note that $p(|h|) = \frac{|h_N|}{|h_T|}$ for $h = h_\circ$.

iv)

$$A = A(t,\tau) = \frac{F(\tau)}{t\,\dot F(\tau)}\,, \qquad r \leqslant t \leqslant R \quad r_* \leqslant \tau \leqslant R_*$$

We note that the function $A(|x|,|h|)$ is well defined and equals $|h|_N$ for $h = h_\circ(x)$.

We proceed to the estimates of the integrand

$$E(Dh) = |Dh|^2 + \Phi(J)\,, \qquad J = J(x,h) = \det Dh$$

We shall explain only basic steps, others being self-explanatory, are left to the reader. Let us begin with

$$\begin{aligned} |Dh|^2 &= |h_N|^2 + |h_T|^2 \geqslant (1-p^2)\,|h_N|^2 \;+\; 2p\,|h_N|\;|h_T| \\ &\geqslant (1-p^2)\,2\,A\;|h|_N \;-\; (1-p^2)\,A^2 \;+\; 2\,p \det Dh \end{aligned} \tag{53}$$

Here we used the pointwise inequalities $|h_N| \geqslant |h|_N$ and $|h_N|\;|h_T| \geqslant \det Dh$, see (66) and (67) to achieve equality here. Next, we use convexity of Φ to estimate the Jacobian term

$$\Phi\big(\det Dh\big) \;\geqslant\; \dot\Phi(b)\det Dh \;+\; \Phi(b) \;-\; b\,\dot\Phi(b) \tag{54}$$

Summing up, we obtain for every $r < t < R$ and $r_* < \tau < R_*$

$$\begin{aligned} E(Dh) \;\geqslant\; & \big[1 - p^2(\tau)\big]\,\frac{2F(\tau)}{\dot F(\tau)}\,\frac{|h|_N}{t} + \\ & + \left[2\,p(\tau) \;+\; \dot\Phi\left(\frac{\tau}{\dot F\,F}\right)\right]\det Dh + \\ & + C(t,\tau) \end{aligned} \tag{55}$$

where

$$C(t,\tau) = \Phi\big(b(\tau)\big) \;-\; b(\tau)\,\dot\Phi\big(b(\tau)\big) - \big[1-p^2(\tau)\big]\,\frac{F^2(\tau)}{t^2\dot F^2(\tau)} \tag{56}$$

Let us emphasize in advance that, upon substitution $t = |x|$ and $\tau = |h(x)|$, the integration of the first two terms will pose no difficulty, as these expressions become free Lagrangians. It remains to estimate from below the expression $C(t,\tau)$, exclusively in terms of t, say

$$C(t,\tau) \geqslant C_\circ(t)\,, \qquad \text{whenever} \;\; r \leqslant t \leqslant R \text{ and } r_* \leqslant \tau \leqslant R_* \tag{57}$$

In this way $C_\circ(|x|)$ will also become a free Lagrangian. But we also want the equality

$$C\big(|x|, H(|x|)\big) \equiv C_\circ\big(|x|\big) \qquad \text{for all} \;\; r \leqslant |x| \leqslant R \tag{58}$$

Now, the choice of $C_\circ(t)$, $r \leqslant t \leqslant R$, is obvious,

$$C_\circ(t) = \inf_{r_* \leqslant \tau \leqslant R_*} C(t,\tau) \tag{59}$$

It leaves us with the task of verifying that the minimum value of the function $\tau \to C(t,\tau)$ is attained at $\tau = H(t)$. Therefore, we must look at the critical points of the function $\tau \to C(t,\tau)$. This involves rather neat computation. With the aid of the radial equilibrium equation (52) for $F = F(\tau)$ we obtain

$$\frac{\partial C(t,\tau)}{\partial \tau} = \frac{F^2(\tau) - t^2}{t^2 \dot{F}^2(\tau)} \, \frac{2\,\tau\,[F(\tau) - \tau \dot{F}(\tau)]^2\,\ddot{\Phi}}{\tau^2\,\ddot{\Phi} + 2F^2(\tau)}\,, \quad \ddot{\Phi} = \ddot{\Phi}\left(\frac{\tau}{F(\tau)\,\dot{F}(\tau)}\right) \tag{60}$$

We refer the interested reader to the appendix for details. We infer from this identity that $C(t,\tau)$ has $\tau = H(t)$ as its only critical point, $F(H(t)) = t$. Furthermore, we see that the function

$$\tau \to C(t,\tau) \quad \text{is} \quad \begin{cases} \text{decreasing for} & r_* \leqslant \tau \leqslant H(t) \\ \text{increasing for} & H(t) \leqslant \tau \leqslant R_* \end{cases} \tag{61}$$

Thus $\tau = H(t)$ is its minimum point, as desired. We then conclude with the sharp inequality

$$C\big(|x|, |h(x)|\big) \geqslant C_\circ\big(|x|\big) \tag{62}$$

for every $h \in \mathscr{F}(\mathbb{X},\mathbb{Y})$. We record for further application that equality occurs in (62) if and only if

$$|h(x)| = H\big(|x|\big) \tag{63}$$

Finally, using the free Lagrangian integral identities (23), (25), (27) and (29) we estimate the energy of h by integrating (55) over the annulus $\mathbb{X}$,

$$\begin{aligned} \mathcal{E}[h] \;\; = \;\; & \int_{\mathbb{X}} E(Dh) \geqslant 2\pi \int_{r_*}^{R_*} [1 - p^2(\tau)]\,\frac{2F(\tau)}{\dot{F}(\tau)}\,\mathrm{d}\tau \, + \\ & + 2\pi \int_{r_*}^{R_*} \left[2\,\tau\,p(\tau) \; + \; \tau \dot{\Phi}\left(\frac{\tau}{F(\tau)\dot{F}(\tau)}\right)\right] \mathrm{d}\tau \, + \\ & + 2\pi \int_r^R t\,C_\circ(t)\,\mathrm{d}t = \mathcal{E}[h^\circ] \end{aligned} \tag{64}$$

The latter is justified because for $t = |x|$ and $\tau = |h(x)|$ we had equality everywhere in the chain of our estimates. It was exactly for this reason how we defined the functions $a(\tau)$, $b(\tau)$, $p(\tau)$ and $A(t,\tau)$. Thus $h_\circ = H\big(|x|\big)\frac{x}{|x|}$ is a minimizer of the energy $\mathcal{E}[h]$ within the class $\mathscr{F}(\mathbb{X},\mathbb{Y})$. For the uniqueness part in Theorem 1, we examine when a mapping $h = h(x)$ in $\mathscr{F}(\mathbb{X},\mathbb{Y})$ gives equality in the chain of estimates that led us to $\mathcal{E}[h] \geqslant \mathcal{E}[h_\circ]$. First, to obtain equality in

(53) it is necessary that $|h_N| = |h|_N$ and $\operatorname{Im} h_T \overline{h_N} = \det Dh = |h_T||h_N|$. The former condition yields

$$\left|h\,\overline{h}_N\right| = |h||h|_N = \frac{1}{2}\left(|h|^2\right)_N = \operatorname{Re} h\overline{h_N} \tag{65}$$

Hence $h\overline{h_N} = \left|h\overline{h}_N\right|$. In conclusion,

$$h(x)\,\overline{h_N(x)} \in \mathbb{R} \qquad\qquad \text{a.e. in } \;\mathbb{X} \tag{66}$$

Similarly $|h_T\overline{h}_N| = |h_T||h_N| = \det Dh = \operatorname{Im}\,\left(h_T\overline{h_N}\right)$; and hence $h_T\overline{h_N} \in i\,\mathbb{R}$. Combined with the condition (66) this yields

$$\overline{h}h_T \in i\,\mathbb{R} \tag{67}$$

For the equality to occur in (54) we need that $\det Dh(x) = b\big(H(|x|)\big)$. In particular,

$$\det Dh(x) \qquad \text{is a function in} \quad |x| \tag{68}$$

Now, it is a simple exercise to verify that conditions (66), (67) and (68) for $h \in \mathscr{F}(\mathbb{X}, \mathbb{Y})$ imply that h is a radial mapping, see [23, Proposition 7.1]. Precisely, we have

$$h(x) = e^{i\alpha}\tilde{H}\big(|x|\big)\,\frac{x}{|x|}\,, \qquad\qquad \text{for some } \;\alpha \in \mathbb{R} \tag{69}$$

where $\tilde{H}$ is a homeomorphism of the interval $[r, R]$ onto $[r_*, R_*]$. We now appeal to the identity (63) which yields $\tilde{H}(|x|) = H(|x|)$. Thus, $h(x) = e^{i\alpha}h_\circ(x)$, completing the proof of Theorem 1 in the elastic case.

4.3 Inelastic case

The idea is much the same as in the elastic case; the only difference is that we shall appeal to the angular free Lagrangian instead of the radial free Lagrangian. However, computation of the term $C(t, \tau)$ is more involved, see the appendix. Here are the neat arrangements of the inequalities

$$\begin{aligned} |Dh|^2 \;\; &= \;\; |h_N|^2 + |h_T|^2 \geqslant (1-q^2)\,|h_T|^2 + 2q\,|h_N|\;|h_T| \\ &\geqslant \;\; (1-q^2)\,2B\,|h_T| \,-\, (1-q^2)\,B^2 \,+\, 2\,q\det Dh \end{aligned} \tag{70}$$

where we choose the parameters q and B to be defined as follows

$$q = q(\tau) = \frac{F(\tau)}{\tau\,\dot{F}(\tau)} < 1 \tag{71}$$

and

$$B = B(t, \tau) = \frac{[H^2(t) - t^2\dot{H}^2(t)]\,\tau\,\dot{F}^2(\tau)}{t\,[\tau^2\,\dot{F}^2(\tau) \,-\, F^2(\tau)]} \tag{72}$$

Note that we have equality in (70) for $h = h^\circ(x)$, $t = |x|$ and $\tau = |h^\circ(x)|$. The Jacobian term is estimated exactly the same way as in (54). We add those inequalities to obtain

$$\begin{aligned} E(Dh) \;\geqslant\; & 2\,\frac{H^2(t) \,-\, t^2\dot{H}^2(t)}{t}\,\mathrm{Im}\,\frac{h_T}{h} + \\ & + \left[\frac{2F(\tau)}{\tau\,\dot{F}\tau} \,+\, \dot{\Phi}\big(b(\tau)\big)\right] \det Dh + \\ & + C(t,\tau) \end{aligned} \tag{73}$$

where $b(\tau) = \frac{\tau}{F(\tau)\,\dot{F}(\tau)}$ and

$$C(t,\tau) = \Phi\big(b(\tau)\big) \,-\, b(\tau)\,\dot{\Phi}\big(b(\tau)\big) - \frac{\Big[H^2(t) \,-\, t^2\dot{H}^2(t)\Big]\;\dot{F}^2(\tau)}{t^2\,\Big[\tau^2\dot{F}^2(\tau) \,-\, F^2(\tau)\Big]} \tag{74}$$

Here we use the inequalities $H^2 - t^2\dot{H}^2 > 0$ and $\frac{|h_T|}{|h|} \geqslant \mathrm{Im}\,\frac{h_T}{h}$. Upon substitution $t = |x|$ and $\tau = |h(x)|$, the first two terms become free Lagrangians so their integrals over $\mathbb{X}$ are independent of $h \in \mathscr{F}(\mathbb{X},\mathbb{Y})$. We now look for the inequality

$$C(t,\tau) \geqslant C_\circ(t)\,, \qquad r \leqslant t \leqslant R\,, \qquad r_* \leqslant \tau \leqslant R_* \tag{75}$$

where we wish to have equality when $\tau = H(t)$. The lengthy computation, with the aid of the radial equilibrium equation (52), reveals that

$$\begin{aligned} \frac{\partial C(t,\tau)}{\partial\tau} \;=\; & \left[\frac{\Big(H^2(t) - t^2\dot{H}(t)\Big)^2}{t^2} - \frac{\Big(F^2(\tau) - \tau^2\dot{F}^2(\tau)\Big)^2}{F^2(\tau)\,\dot{F}^4(\tau)}\right] \times \\ & \times \frac{2\Big[F(\tau) - \tau\dot{F}(\tau)\Big]^2\;\dot{F}^2(\tau)\,\tau\,\ddot{\Phi}}{\Big[F^2(\tau) \,-\, \tau^2\dot{F}^2(\tau)\Big]^2\,\Big[\tau^2\ddot{\Phi} \,+\, 2F^2(\tau)\Big]}\,, \quad \ddot{\Phi} = \ddot{\Phi}\left(\frac{\tau}{F(\tau)\,\dot{F}\tau}\right) \end{aligned} \tag{76}$$

Here we notice that the first factor vanishes when $\tau = H(t)$, while the second factor is always positive. We need only examine the sign of the first factor in order to conclude that $\tau = H(t)$ is a minimum point of the function $\tau \to C(t,\tau)$. This factor takes the form

$$\Gamma(t) \,-\, \Gamma\big(F(\tau)\big) \tag{77}$$

where Γ is defied by

$$\Gamma(t) = \frac{\Big[H^2(t) \,-\, t^2\dot{H}^2(t)\Big]^2}{t^2}\,, \qquad r \leqslant t \leqslant R \tag{78}$$

We point out that $\Gamma(t)$ is decreasing in the inelastic case. Indeed, we have

$$\dot{\Gamma}(t) = 2\left[(H - t\dot{H})^2 + 2t^3 \dot{H}\,\ddot{H}\right] \frac{(t^2\dot{H}^2 - H^2)}{t^3} < 0 \tag{79}$$

because $0 < t\dot{H} < H$ and $\ddot{H} > 0$. This shows that

$$\Gamma(t) - \Gamma\big(F(\tau)\big) \quad \text{is} \quad \begin{cases} \text{negative if} & r_* \leqslant \tau < H(t) \\ \text{zero if} & \tau = H(t) \\ \text{positive if} & H(t) < \tau \leqslant R_* \end{cases} \tag{80}$$

Consequently, the function $\tau \to C(t,\tau)$ is decreasing for $r_* < \tau \leqslant H(t)$ and increasing for $H(t) \leqslant \tau \leqslant R_*$. Thus $\tau = H(t)$ is its minimum point, as desired. As a matter of fact $\tau = H(t)$ is the only minimum point of the function $\tau \to C(t,\tau)$. This fact is crucial in establishing the uniqueness part of Theorem 1, as in the elastic case, though we shall not go into details in this case.

Finally, integrating (73) over the annulus we conclude that

$$\begin{aligned} \mathcal{E}[h] \quad &= \quad \int_{\mathbb{X}} E(Dh) \geqslant 4\pi \int_r^R \left[H^2(t) - t^2\dot{H}^2(t)\right] \mathrm{d}t \\ &\quad + 2\pi \int_{r_*}^{R_*} \left[\frac{2F(\tau)}{\dot{F}(\tau)} + \dot{\Phi}\big(b(\tau)\big)\right] \mathrm{d}\tau \\ &\quad + 2\pi \int_r^R t\, C_\circ(t)\,\mathrm{d}t = \int_{\mathbb{X}} E(Dh_\circ) = \mathcal{E}[h_\circ] \end{aligned} \tag{81}$$

We have equality when $h(x) = H\big(|x|\big)\frac{x}{|x|}$. The arguments for the uniqueness are much the same as in the elastic case. A slight difference is that we first derive (67), because $\frac{|h_T|}{|h|} \geqslant \operatorname{Im} \frac{h_T}{h}$ must hold as equality. Then we obtain (66) from the additional requirement that $|h_T||h_N| = \operatorname{Im}\big(h_T\overline{h_N}\big)$. The rest of the proof is the same.

5 Appendix

5.1 A computation for $C_\tau(t,\tau)$ in elastic case.

We rewrite the formula (56) as,

$$C(t,\tau) = \Phi\big(b(\tau)\big) - b(\tau)\,\dot{\Phi}\big(b(\tau)\big) - \left(\frac{F^2(\tau)}{\dot{F}^2(\tau)} - \tau^2\right)\frac{1}{t^2}, \quad b(\tau) = \frac{\tau}{F(\tau)\,\dot{F}(\tau)} \tag{82}$$

Direct differentiation with respect to the τ-variable yields

$$\frac{\partial C(t,\tau)}{\partial \tau} \quad = \quad \left(\tau^2\ddot{\Phi} + \frac{2F^4}{t^2}\right)\frac{\ddot{F}}{F^2\dot{F}^3} + \frac{\tau(\tau\dot{F} - F)}{F^3\dot{F}^2}\ddot{\Phi} + \frac{2(\tau\dot{F} - F)}{t^2\dot{F}} \tag{83}$$

where $\ddot{\Phi} = \ddot{\Phi}\left(\frac{\tau}{F\dot{F}}\right)$. Then we substitute the right hand side of (52) to conclude with the formula (60).

5.2 A computation for $C_\tau(t,\tau)$ in inelastic case.

First, taking the partial derivative of C (C is given by the formula (74)) with respect the variable τ, we have

$$\begin{aligned} C_\tau(t,\tau) &= \left[\frac{\tau^2}{F^3\dot{F}} + \frac{\tau^2\ddot{F}}{F^2\dot{F}^3} - \frac{\tau}{F^2\dot{F}^2}\right]\ddot{\Phi}(b(\tau)) + \\ & \quad - \frac{2\left(H^2 - t^2\dot{H}^2\right)^2}{t^2}\left[\frac{F\dot{F}^3 - F^2\dot{F}\ddot{F} - \tau\dot{F}^4}{\left(\tau^2\dot{F}^2 - F^2\right)^2}\right] \end{aligned} \tag{84}$$

Second, we rewrite the radial equilibrium equation (52) as

$$\left[\frac{\tau^2}{F^3\dot{F}} + \frac{\tau^2\ddot{F}}{F^2\dot{F}^3} - \frac{\tau}{F^2\dot{F}^2}\right]\ddot{\Phi}(b(\tau)) = \frac{2\left(\dot{F}^2\tau^2 - F^2\right)^2}{F^2\,\dot{F}^4(\tau)}\left[\frac{F\dot{F}^3 - F^2\dot{F}\ddot{F} - \tau\dot{F}^4}{\left(\tau^2\dot{F}^2 - F^2\right)^2}\right]$$

Then we substitute it into (84) to obtain

$$C_\tau(t,\tau) = \left[\frac{2\left(\dot{F}^2\tau^2 - F^2\right)^2}{F^2\,\dot{F}^4} - \frac{2\left(H^2 - t^2\dot{H}^2\right)^2}{t^2}\right]\frac{F\dot{F}^3 - F^2\dot{F}\ddot{F} - \tau\dot{F}^4}{\left(\tau^2\dot{F}^2 - F^2\right)^2} \tag{85}$$

We appeal to the radial equilibrium equation (52) again to eliminate $\ddot{F}$. Then the numerator in the last term can be factored as

$$F\dot{F}^3 - F^2\dot{F}\ddot{F} - \tau\dot{F}^4 = -\left(F - \tau\dot{F}\right)^2\frac{\tau\,\ddot{\Phi}}{\tau^2\ddot{\Phi} + 2F^2} \tag{86}$$

Substitute this into (85) we obtain the formula (76) for $C_\tau(t,\tau)$.

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