QUASICONFORMAL HYPERELASTICITY
WHEN
CAVITATION IS NOT ALLOWED

Tadeusz Iwaniec *Jani Onninen*[*]

Abstract

This paper features a class of mappings $h = (h^1, ..., h^n) : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ between domains $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ having finite quasiconformal energy. The associated energy is defined for mappings in the Sobolev space $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$ by the rule:

$$\mathscr{E}[h] = \int_{\mathbb{X}} \mathbf{E}(x, h, Dh)\, \mathrm{d}x\,, \qquad Dh = \left[\frac{\partial h^i}{\partial x_j}\right] \in \mathbb{R}^{n\times n} \tag{1}$$

The class of energy integrals we consider here is invariant under quasiconformal change of the variables in the reference configuration $\mathbb{X} \subset \mathbb{R}^n$. This motivates our calling the subject matter *quasiconformal hyperelasticity.* A model example is the n-harmonic integral, also referred to as conformal energy,

$$\mathscr{E}[h] = \int_{\mathbb{X}} \|\, Dh(x)\, \|^{\,n}\, \mathrm{d}x\,, \qquad \|\, Dh\, \|^{\,2} = \mathrm{Tr}\,(D^*h\, Dh) \tag{2}$$

For the general integrand $\mathbf{E} : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{n\times n} \to \mathbb{R}$ we assume here that

$$\alpha \,\|\, X \,\|^{\,n} \leqslant \mathbf{E}(x, y, X) \leqslant \beta \,\|\, X \,\|^{\,n}\,, \qquad 0 < \alpha \leqslant \beta < \infty \tag{3}$$

A natural question, both from the theoretical and practical points of view, is whether or not the domains $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ admit homeomorphisms $h = (h^1, ..., h^n) : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of finite energy and, if so, whether or not the energy integral assumes its minimum value among all homeomorphisms $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in a given homotopy class. We adopt the language and various

---

[*] Iwaniec was supported by the National Science Foundation grants DMS-0301582 and DMS-0244297, and Onninen by the National Science Foundation grants DMS-0400611 and DMS-0632409.

AMS (2000) Classification. Primary 30C60, 35J15, 35J70
Keywords n-Harmonics, Extremal Problems, Quasiconformal Mappings, Variational Integrals

interpretations of elasticity theory where a large part of our motivation has originated. We shall see that the lower bound at (3) prevents all sorts of cavitation from growing to higher dimensions. For instance, it will be impossible to open a puncture in a solid body into a hole. As a concrete example, the map $y = h(x) \overset{\text{def}}{=\!=} x + \frac{x}{|x|}$, [5], has infinite conformal energy near the origin, as it takes the punctured space into the complement of the unit ball. We shall see, by rather sophisticated topological arguments, that quasiconformal hyperelasticity prevents formation of discontinuities along flat fractures $\mathfrak{X} \subset \mathbb{X}$ of dimension $\dim \mathfrak{X} \leqslant n-2$. Even more, $\dim \mathfrak{X}$ cannot increase. The concept of *total conformal energy* is considered, generalizing the theory of quasiconformal mappings. We show that the deformations of finite total energy extend homeomorphically to the internal fracture $\mathfrak{X} \subset \mathbb{X}$ if $\dim \mathfrak{X} \leqslant n-2$. The p-harmonic integrals on punctured balls are also considered, $1 \leqslant p < n$. Surprisingly, homeomorphisms having smallest p-harmonic energy, need not be radially symmetric.

These topics grew out of our study of the variational integrals in nonlinear elasticity but also as a synthesis of recent advances in the multidimensional quasiconformal geometry, mappings of unbounded distortion in particular. Interplay of geometric and topological methods is critical in all of our proofs.

Contents

1 Introduction

1.1 Overview

Although Riemannian manifolds are not the domains of our study here the ideas and various generalizations are naturally found in a differential-geometric setting. Let $\mathbb{X}$ and $\mathbb{Y}$ be Riemannian n -manifolds $n \geqslant 2$. No explicit mention of the orientation and the metric tensors will be needed. However, we reserve once and for always the induced volume forms $\mathrm{d}x \in \wedge^n \mathbb{X}$ and $\mathrm{d}y \in \wedge^n \mathbb{Y}$, for integration purposes. The main theme is about mappings $h : \mathbb{X} \to \mathbb{Y}$ of the Sobolev class $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$. The reader should be aware that smooth mappings from $\mathbb{X}$ into $\mathbb{Y}$ are dense in $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$ [1], see [69, 70, 15, 16]. The linear tangent map $Dh(x) : \mathbf{T}_x\mathbb{X} \longrightarrow \mathbf{T}_y\mathbb{Y}$ is well defined at almost every point $x \in \mathbb{X}$, where $y = h(x)$. A natural question is whether or not the n-harmonic integral

$$\mathscr{E}[h] = \int_{\mathbb{X}} \| Dh(x) \|^n \, \mathrm{d}x \tag{4}$$

is finite and, if so, whether or not it assumes its minimum value among all homeomorphisms $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in a given homotopy class. Here we use the Hilbert-Schmidt norm of the tangent map $Dh : \mathbf{T}\mathbb{X} \to \mathbf{T}\mathbb{Y}$,

$$\| Dh \|^2 = \operatorname{Tr} (D^*h \cdot Dh) \tag{5}$$

where $D^*h : \mathbf{T}\mathbb{X} \to \mathbf{T}\mathbb{Y}$, stands for the transpose tangent map. In dimension $n = 2$ the Dirichlet integral in (4) is central in the theory of harmonic maps [22]. Because of conformal invariance we call $\mathscr{E}[h]$ the conformal energy of h or, sometimes, n-harmonic energy. In higher dimensions, the n-harmonic alternative to the classical Dirichlet integral has drawn the attention of researchers in Geometric Function Theory [67, 10, 32, 76, 68, 39]. Homeomorphisms which minimize the conformal energy are referred to as *n-harmonic deformations.*

There are several natural ways that n-harmonic deformations arise in analysis. One of them is the mathematical model of nonlinear elasticity pioneered by S.S. Antman [3], J.M. Ball [5], and P.G. Ciarlet [19]. This theory is concerned with elastic deformations $h : \mathbb{X} \to \mathbb{Y}$ of a material body in $\mathbb{X} \subset \mathbb{R}^n$, called the *reference configuration*, onto a given domain $\mathbb{Y} \subset \mathbb{R}^n$, called the *deformed configuration.* Hyperelastic materials are the ones that possess a stored-energy function

$$\mathbf{E} : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{n \times n} \to \mathbb{R} \tag{6}$$

---

[1] When $p < n$, the topology can inhibit the space $\mathscr{C}^\infty(\mathbb{X}, \mathbb{Y})$ from being dense in $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, see [9, 30, 28].

The associated energy of h, defined in a suitable class of weakly differentiable mappings $h : \mathbb{X} \to \mathbb{Y}$, is given by:

$$\mathscr{E}[h] = \int_{\mathbb{X}} \mathbf{E}(x, h, Dh)\, \mathrm{d}x \tag{7}$$

In this theory the Jacobian matrix $Dh(x) \in \mathbb{R}^{n\times n}$ is often referred to as the deformation gradient. By virtue of the principle of non-penetration of matter the final goal is to find an injective map $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ with the smallest energy.

In another direction, we recall Geometric Function Theory in $\mathbb{R}^n$ and the governing variational integrals. At almost every point $x \in \mathbb{X}$, we have well defined differential matrix $Dh = [\,\frac{\partial h^i}{\partial x_j}\,] \in \mathbb{R}^{n\times n}$ and its Jacobian determinant $J(x,h) = \det Dh(x)$. All mappings considered have nonnegative Jacobian.

A mapping $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is conformal at $x \in \mathbb{X}$ if $\|\, Dh(x) \,\|^{\,n} = n^{\frac{n}{2}} J(x,h)$. This is equivalent to the nonlinear Cauchy-Riemann system of PDEs;

$$D^* h(x) \cdot Dh(x) = J(x,h)^{\frac{2}{n}} \mathbf{I} \tag{8}$$

It is evident that the n-harmonic energy of conformal deformations $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ does not depend on the choice of the deformation,

$$\mathscr{E}[h] = \int_{\mathbb{X}} \|\, Dh(x) \,\|^{\,n}\, \mathrm{d}x \;=\; n^{\frac{n}{2}} \int_{\mathbb{X}} J(x,h)\, \mathrm{d}x \;=\; n^{\frac{n}{2}} |\mathbb{Y}| \tag{9}$$

However, for more general homeomorphisms $g : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in the Sobolev space $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$, we have

$$\mathscr{E}[g] \geqslant \|\, Dg(x) \,\|^{\,n} \;\geqslant n^{\frac{n}{2}} \int_{\mathbb{X}} J(x,g)\, \mathrm{d}x = n^{\frac{n}{2}} |\mathbb{Y}| \qquad \text{-the absolute minimum}$$

In other words, conformal deformations $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ are characterized as having absolutely the smallest n-harmonic energy. However, it is rare in higher dimensions that two topologically equivalent manifolds are conformally equivalent, because of Liouville's rigidity theorem [52]. Even in the plane, by virtue of Schottky's Theory, [71], multiply connected domains need not be conformally equivalent. Quasiconformal theory offers more mappings.

A homeomorphism $g : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in the Sobolev space $\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{X}, \mathbb{Y})$ is said to be K-quasiconformal, $1 \leqslant K < \infty$, if

$$\|\, Dg(x) \,\|^{\,n} \leqslant n^{\frac{n}{2}}\, K\, J(x,g)$$

Naturally, quasiconformal mappings belong to the Sobolev space $\mathscr{W}^{1,n}_{\mathrm{loc}}(\mathbb{X},\mathbb{Y})$ because the Jacobian of a homeomorphism $\mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{X},\mathbb{R}^n)$ is always locally integrable [24]. It is again striking how tight the relations are between quasiconformal theory and material sciences; in particular, nonlinear elasticity, microstructures and crystals [6, 7, 20, 63]. In continuum mechanics, the positive definite matrix $\mathbf{C}(x) = D^*h(x)\, Dh(x)$ is referred to as the *right Cauchy-Green* deformation tensor. While on the other hand, there is a fundamental interplay between quasiconformal mappings and the *Beltrami equation*

$$D^*h(x)\; Dh(x) \;\overset{\mathrm{def}}{=\!=}\; J(x,h)^{\frac{2}{n}}\; \mathbf{G}(x)\;, \qquad \det \mathbf{G}(x) \equiv 1 \tag{10}$$

where $\mathbf{G} = \mathbf{G}(x)$, called the *distortion tensor* of h, is none other than the Cauchy-Green tensor normalized so as to have determinant identically equal to one, [10, 31, 67, 68]. Now, the matrix function $\mathbf{G} = \mathbf{G}(x) = [G_{ij}(x)] \in \mathbb{R}^{n\times n}$ itself can be viewed as a Riemannian metric on $\mathbb{X}$. In this way h becomes conformal with respect to this (usually only measurable) uniformly elliptic metric structure on $\mathbb{X}$. Slightly more general first order systems of PDEs in this theory arise when mappings are view as conformal with respect to suitable measurable Riemannian structures on both $\mathbb{X}$ and $\mathbb{Y}$, [10, 38, 36]. The general Beltrami systems, though almost a tautology, are very useful:

$$D^*h(x)\cdot \mathbf{H}(h)\cdot Dh(x) \;\overset{\mathrm{def}}{=\!=}\; J(x,h)^{\frac{2}{n}}\; \mathbf{G}(x)\;, \qquad \det \mathbf{G}(x) = \det \mathbf{H}(y) \equiv 1 \tag{11}$$

It is in this Riemannian manifold framework that variational interpretations of quasiconformal mappings really crystalize. For example, the solutions to the Beltrami equation (11) are absolute minimizers of their own energy integral. Indeed, a homeomorphism $h:\mathbb{X}\to\mathbb{Y}$ of Sobolev class $\mathscr{W}^{1,n}(\mathbb{X},\mathbb{Y})$ solves the Beltrami equation (11) if and only if

$$\mathscr{E}[h] \;\overset{\mathrm{def}}{=\!=}\; \int_{\mathbb{X}} \mathbf{E}(x,h,Dh)\,\mathrm{d}x \;=\; n^{\frac{n}{2}} \int_{\mathbb{X}} J(x,h)\,\mathrm{d}x \;=\; n^{\frac{n}{2}}\,|\mathbb{Y}| \tag{12}$$

where the integrand is defined on $\mathbb{X}\times\mathbb{Y}\times\mathbb{R}^{n\times n}$ by the rule

$$\mathbf{E}(x,y,X) = \;\left[\mathrm{Tr}\,(\mathbf{G}^{-1}\, X^*\mathbf{H}X)\right]^{\frac{n}{2}}\;, \quad \mathbf{G}^{-1} = \mathbf{G}^{-1}(x) \text{ and } \mathbf{H} = \mathbf{H}(y)$$

It has been increasingly acknowledged that in higher dimensions the n-harmonic deformations in $\mathbb{R}^n$ are a useful generalization of the quasiconformal mappings. One of the most appealing recent discoveries is a connection between the n-harmonic energy of $h:\mathbb{X}\to\mathbb{Y}$ and the inner distortion function of the inverse mapping $f = h^{-1}:\mathbb{Y}\to\mathbb{X}$, see [4, 33, 34, 65]. For this we recall the matrix

$D^\sharp f \in \mathbb{R}^{n\times n}$ of $(n-1)\times(n-1)$- cofactors. The differential matrix, its determinant and the cofactors are the building blocks of polyconvex distortion functions [39]. For a mapping $f \in \mathscr{W}^{1,1}_{\textrm{loc}}(\mathbb{Y},\mathbb{X})$ of finite distortion we may consider the *inner distortion* $K_{_I}(y) = K_{_I}(y,f) \geqslant 1$, given by

$$K_{_I}(y,f) = \frac{\|\, D^\sharp f(y)\,\|^{\,n}}{n^{\frac{n}{2}} J(y,f)^{n-1}}\,, \qquad K_{_I}(y,f) = 1 \ \text{ if } \ J(x,f) = 0$$

The pullback of the n-form $K_{_I}(y)\mathrm{d}y \in \wedge^n\mathbb{Y}$ by the mapping $h : \mathbb{X} \to \mathbb{Y}$ is equal to $\|\,Dh(x)\,\|^{\,n}\mathrm{d}x \in \wedge^n\mathbb{X}$. This observation is the key to the following fundamental identity, see [4, 33, 34, 65, 21].

Proposition 1.1. (TRANSITION TO THE INVERSE) *Let* $f \in \mathscr{W}^{1,n-1}_{\textrm{loc}}(\mathbb{Y},\mathbb{X})$ *be a homeomorphism of integrable inner distortion between domains* $\mathbb{Y},\mathbb{X} \subset \mathbb{R}^n$. *Then the inverse map* $h = f^{-1} : \mathbb{X} \xrightarrow{onto} \mathbb{Y}$ *belongs to the Sobolev class* $\mathscr{W}^{1,n}(\mathbb{X},\mathbb{Y})$, *and we have*

$$n^{\frac{n}{2}} \int_{\mathbb{Y}} K_{_I}(y,f)\,dy \quad = \quad \int_{\mathbb{X}} \|\,Dh(x)\,\|^{\,n}\,dx \tag{13}$$

From yet another perspective, classical Teichmüller theory is concerned, broadly speaking, with extremal mappings between Riemann surfaces. The Teichmüller mapping is exactly the one whose distortion function has the smallest possible supremum norm. The existence and uniqueness of such an extremal quasiconformal map within a given homotopy class is the heart of Teichmüller's theory. Now, in view of the identity (13), minimizing the $\mathscr{L}^1$-norm of the inner distortion amounts to the study of n-harmonic mappings. Is there any better motivation for a little theory of qusiconformal hyperelasticity?

1.2 Statement of results

THE n-HARMONIC ENERGY. The radial stretching $h(x) = (1+|x|)\frac{x}{|x|}$, introduced by J.M. Ball [5] is the basic example of cavitation. This map takes the punctured space $\mathbb{R}^n_\circ = \mathbb{R}^n \setminus \{0\}$ outside the unit ball, a hole in $\mathbb{R}^n$. Note that the conformal energy of h near the origin is infinite. As a matter of fact making a hole from a puncture always requires infinite conformal energy.

Proposition 1.2. (REMOVABILITY OF PUNCTURES) *Let* $x_\circ \in \mathbb{X}$ *be a point in a domain* $\mathbb{X} \subset \mathbb{R}^n$ *and* $h : \mathbb{X} \setminus \{x_\circ\} \to \mathbb{R}^n$ *a continuous injective map having finite conformal energy. Then* h *extends to* $\mathbb{X}$ *as a homeomorphism in the Sobolev class* $\mathscr{W}^{1,n}(\mathbb{X},\mathbb{R}^n)$.

A natural question arises if the finite energy condition prevents the deformations from having discontinuities along k-dimensional cracks and fractures in $\mathbb{X}$.

In dimension $n=2$ the conformal mapping theory provides us with deformations of finite (actually minimal) Dirichlet integral. Thus, even the extremal deformations of smallest conformal energy may create a hole out of a 1-dimensional crack; the inverse of the Zhukovsky conformal map $f(z)=\frac{1}{2}\left(z+\frac{1}{z}\right)$ serves well as an example. As a matter of fact for every $n \geqslant 2$ it is possible to make a hole out of a crack of dimension $k=n-1$,

Example 1.1. (A CAVITY) Let $\mathfrak{X}$ be an $(n-1)$-dimensional disk in $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ of the form $\mathfrak{X} = \{(w,0) \, : \, |w| \leqslant 1\}$. Define a $\mathscr{C}^\infty$-smooth homeomorphism $h : \mathbb{R}^n \setminus \mathfrak{X} \to \mathbb{R}^n$ by the rule:

$$h(x) = h(w,t) = \begin{cases} (w, \; t+\tau(w)) & \text{if} \quad t>0 \\ (w, \; t) & \text{if} \quad t=0 \; \text{ and } \; |w|>1 \end{cases}$$

where $\tau \in \mathscr{C}^\infty_\circ(\mathbb{R}^{n-1})$ is positive for $|w|<1$ and vanishes for $|w| \geqslant 1$. This map has bounded gradient and yet makes a hole.

The situation is dramatically different with lower dimensional cracks. In order to state our results in full generality we need to introduce the notion of quasiconformally k-flat cracks. This notion is very natural since the property of having finite n-harmonic energy is invariant under quasiconformal change of the independent variables.

Definition 1.1. A closed set $\mathfrak{X} \subset \mathbb{R}^n$ is said to be quasiconformally k-flat, $1 \leqslant k \leqslant n-1$, if every point in $\mathfrak{X}$ has a neighborhood $\mathbb{U} \subset \mathbb{R}^n$ such that $\varphi(\mathfrak{X} \cap \mathbb{U}) \subset \mathbb{R}^k$, for some quasiconformal map $\varphi : \mathbb{U} \to \mathbb{R}^n$.

The following is the key result of this paper.

THEOREM 1.1. *Let $\mathbb{X}$ be a bounded domain in $\mathbb{R}^n$ and $\mathfrak{X}$ a closed quasiconformally k-flat subset of $\mathbb{X}$, $1 \leqslant k \leqslant n-2$. Then every continuous injection $h : \mathbb{X} \setminus \mathfrak{X} \to \mathbb{R}^n$ of Sobolev class $\mathscr{W}^{1,n}(\mathbb{X} \setminus \mathfrak{X}, \mathbb{R}^n)$ extends continuously to $\mathbb{X}$. This extension belongs to $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$ and satisfies Lusin's condition $\mathscr{N}$.*

The proof is based on rather involved topological arguments which we develop throughout Sections 2-6. However, it is much easier to deal with the question of continuity under suitable assumptions about the image of the crack. As for the crack itself, we need only assume that it is relatively closed and has zero measure. Let us denote by $h\{\mathfrak{X}\}$ the cluster set of h over $\mathfrak{X}$, see Section 4.1.

THEOREM 1.2. *Let $\mathfrak{X}$ be a relatively closed subset of zero measure in a bounded domain $\mathbb{X} \subset \mathbb{R}^n$. Suppose that we are given a bounded mapping $h \in$*

$\mathscr{W}^{1,n}(\mathbb{X},\mathbb{R}^n)$ *which is a continuous injection outside* $\mathfrak{X}$*, and that the cluster set* $h\{\mathfrak{X}\}$ *does not separate domains in* $\mathbb{R}^n$*. Then the map* $h : \mathbb{X} \setminus \mathfrak{X} \to \mathbb{R}^n$ *extends continuously to* $\mathbb{X}$ *and satisfies Lusin's condition* $\mathscr{N}$*.*

Remark 1.1. We say that a closed set Υ does not separate domains in $\mathbb{R}^n$ if for every open connected set $\mathbb{V} \subset \mathbb{R}^n$ the set $\mathbb{V} \setminus \Upsilon$ is still connected. For instance, this property is guaranteed if $\dim \Upsilon \leqslant n-2$, see Theorem 3.3. We also remark that under the hypothesis of Theorem 1.1 or Theorem 1.2 the map $h : \mathbb{X} \to \mathbb{R}^n$ need be neither discrete nor open. Moreover, $h(\mathbb{X})$ need not be a domain unless $\mathfrak{X}$ is compact. We shall refer to a compact crack as *internal fracture* in $\mathbb{X}$.

Having established continuity of $h : \mathbb{X} \to \mathbb{R}^n$ our next result is topological in nature, interesting in its own. For this we recall:

Definition 1.2. A flat k-cell in $\mathbb{R}^n$ is a continuum $\mathfrak{X} \subset \mathbb{R}^n$ of the form $\mathfrak{X} = \varphi(\mathbb{D}^k)$, where $\mathbb{D}^k$ stands for the unit disk of dimension k,

$$\mathbb{D}^k = \{(x_1, ..., x_k, 0, ..., 0)\,;\;\; x_1^2 + ...x_k^2 \leqslant 1\} \subset \mathbb{R}^k \subset \mathbb{R}^n$$

and φ is a homeomorphism of the entire space $\mathbb{R}^n$ onto itself. We write $k = \dim \mathfrak{X}$. If, moreover, φ is quasiconformal then we call $\mathfrak{X}$ quasiconformal k-cell.

THEOREM 1.3. *Let* $\mathbb{X}$ *be a bounded domain in* $\mathbb{R}^n$ *and* $\mathfrak{X} \subset \mathbb{X}$ *a* k*-cell,* $1 \leqslant k \leqslant n-2$*. Suppose we are given a continuous map* $h : \mathbb{X} \to \mathbb{R}^n$ *which is injective on* $\mathbb{X} \setminus \mathfrak{X}$*. Then* $\mathbb{Y} = h(\mathbb{X})$ *is a domain of the same topological type as* $\mathbb{X}$ *and* $\Upsilon = h(\mathfrak{X})$ *is a continuum in* $\mathbb{Y}$*. Moreover, there are no* k*-separators in* $\mathbb{R}^n \setminus \mathbb{V}$*, where* $\mathbb{V}$ *stands for the unbounded component of* $\mathbb{R}^n \setminus \Upsilon$*. In particular* Υ *does not contain any flat* $(k+1)$*-cell.*

Here the term k*-separator*, $1 \leqslant k \leqslant n-1$, referees to a set in $\mathbb{R}^n$ of the form $\Sigma = \psi(\mathbb{K}) \subset \mathbb{R}^n$, where $\mathbb{K} \subset \mathbb{R}^{k+1} \subset \mathbb{R}^n$ is a continuum separating $\mathbb{R}^{k+1}$ and ψ is a homeomorphism of $\mathbb{R}^n$ onto itself. The classical examples of k-separators are the unknotted k-spheres.

Theorem 1.3 along with Theorem 1.1 yield some information about the structure of the image of the cracks under mappings of finite energy. We summarize it as:

THEOREM 1.4. *Let* $\mathbb{X}$ *be a bounded domain in* $\mathbb{R}^n$ *and* $\mathfrak{X} \subset \mathbb{X}$ *a quasiconformal* k*-cell,* $1 \leqslant k \leqslant n-2$*. Suppose we are given a homeomorphism* $h : \mathbb{X} \setminus \mathfrak{X} \to \mathbb{R}^n$ *of finite conformal energy. Then* h *extends continuously to* $\mathbb{X}$ *and the image* $\mathbb{Y} = h(\mathbb{X})$ *is a domain of the same topological type as* $\mathbb{X}$*. Moreover, the crack* $\Upsilon = h(\mathfrak{X})$ *does not contain any unknotted* k*-sphere. In particular,* Υ *contains no flat* $(k+1)$*-disks.*

Remark 1.2. In general $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ need not be an injection, not even an open map, see the example in Section 9.

One particular consequence of Theorem 1.4 merits a separate statement.

Corollary 1.1. *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be balls in* $\mathbb{R}^n$, $\mathfrak{X} \subset \mathbb{X}$ *and* $\Upsilon \subset \mathbb{Y}$ *closed disks of dimension* $1 \leqslant \dim \mathfrak{X} < \dim \Upsilon \leqslant n-1$, *and* $h : \mathbb{X} \setminus \mathfrak{X} \xrightarrow{onto} \mathbb{Y} \setminus \Upsilon$ *a homeomorphism. Then*

$$\int_{\mathbb{X}} \| Dh(x) \|^n \, \mathrm{d}x \; = \; \infty$$

Theorems 1.1- 1.4 will be established via delicate interplay between oscillation estimates of homeomorphisms with finite energy and some topological results, such as: Strong Jordan Separation Theorem recently established by Lafont [50] and a generalization of Borsuk-Ulam Antipodal Theorem due to Joshi [42]. These and other topological prerequisites are stated in the next section.

We shall commence with the simple case when $\mathfrak{X} = \{x_\circ\}$ is a puncture, see Proposition 1.2. This rather elementary proposition paves the way for far reaching generalizations, as presented in Theorems 1.1- 1.2.

The p-harmonic energy. When speaking of finite energy an obvious dual question to ask is whether the infimum of the energy in a given class of deformations $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is attained. In particular, it cannot be zero. The latter is always true for quasiconformal hyperelastic materials. Indeed, we have

$$\mathcal{E}[h] = \int_{\mathbb{X}} \mathbf{E}(x, h, Dh) \, \mathrm{d}x \geqslant C \int_{\mathbb{X}} J(x,h) \, \mathrm{d}x = C \, |\mathbb{Y}| \neq 0$$

We now forsake for a while the conformal energy and take on stage the p-harmonic integrals

$$\mathcal{E}_p[h] = \int_{\mathbb{X}} \| Dh(x) \|^p \, \mathrm{d}x \qquad \text{with } \; 1 \leqslant p < n \tag{14}$$

The infimum energy remains positive for $p > n-1$, provided each of the domains $\mathbb{X}$ and $\mathbb{Y}$ have at least two boundary components. This is in marked contrast with the theory of p-capacity where the range of the functions $h : \mathbb{X} \xrightarrow{\text{onto}} [0,1]$ is one dimensional; the infimum is attained if and only if it is positive. We demonstrate various differences and phenomena for n-dimensional deformations of punctured balls. When $n = 2$ then the $(n-1)$-harmonic energy of mappings $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}$ with $h(0) = 0$, stays bounded away from zero. This is not the case for $n \geqslant 3$.

THEOREM 1.5. *Let $\mathcal{F}$ denote the class of all homeomorphisms $h : \mathbb{B} \to \mathbb{B}$ in the Sobolev space $\mathscr{W}^{1,p}(\mathbb{B}\,,\,\mathbb{B})$ such that $h(0) = 0$. Then*

$$\inf_{h\in\mathcal{F}} \int_{\mathbb{B}} \|\, Dh(x)\,\|^{\,p}\, \mathrm{d}x = \begin{cases} 0\,, & \textit{if } \ p \leqslant n-1 \ \textit{and} \ n = 3,4,... \\ \textit{positive}\,, & \textit{otherwise} \end{cases} \tag{15}$$

It is surprising that the zero infimum energy here cannot occur within spherically symmetric mappings if the dimension is greater than 2, see Theorem 10.2.

The total n-harmonic energy. We note that both a quasiconformal mapping and its inverse always finite conformal energy, but not conversely. It is well known that no quasiconformal deformation can open an inward spike pointing into the ball [27]. On the other hand, a homeomorphism with finite n-harmonic energy can produce inward spikes. From this point of view n-harmonic deformations seem to be more general and, as such, better suited to the mathematical models of the elasticity theory. However, if one wishes not to go far away from the classical quasiconformal theory, then the right concept is the *total energy*

$$\mathcal{E}[h] = \int_{\mathbb{X}} \big[\, \|\, Dh(x)\,\|^{\,n} \ + n^{\frac{n}{2}} K_{_I}(x,h)\big] \ \mathrm{d}x < \infty \tag{16}$$

It yields finite conformal energy of the inverse map $\ f = h^{-1} : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ as well. Indeed, in view of the identity (13) this integral can expressed as

$$\mathcal{E}[h,f] = \int_{\mathbb{X}} \|\, Dh(x)\,\|^{\,n}\, \mathrm{d}x + \int_{\mathbb{Y}} \|\, Df(y)\,\|^{\,n}\, \mathrm{d}y \tag{17}$$

Now, combining Theorems 1.1 and 1.2, we obtain

Corollary 1.2. *Let $\mathbb{X}$ be bounded domain in $\mathbb{R}^n$. Suppose that $\mathfrak{X}$ is a compact quasiconformally k-flat subset of $\mathbb{X}$, $1 \leqslant k \leqslant n-2$. Then every homeomorphism $h : \mathbb{X} \setminus \mathfrak{X} \to \mathbb{R}^n$ of finite total energy*

$$\mathcal{E}[h] = \int_{\mathbb{X}} \big[\, \|\, Dh(x)\,\|^{\,n} \ + n^{\frac{n}{2}} K_{_I}(x,h)\big] \ \mathrm{d}x < \infty \tag{18}$$

extends as a homeomorphism on $\mathbb{X}$.

While, applying Theorem 1.4 yields

Corollary 1.3. *Let $\mathfrak{X} \subset \mathbb{X}$ and $\Upsilon \subset \mathbb{Y}$ be quasiconformal cells in bounded domains $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$. If there is a homeomorphism $\ h : \mathbb{X} \setminus \mathfrak{X} \xrightarrow{onto} \mathbb{Y} \setminus \Upsilon$ of finite total energy, then* $\dim \mathfrak{X} = \dim \Upsilon$*, provided* $1 \leqslant \dim \mathfrak{X} \leqslant n-2$ *or, equivalently,* $1 \leqslant \dim \Upsilon \leqslant n-2$.

Yet another form of the total energy, the average of the distortion of h and its inverse f, merits mentioning here.

$$\mathcal{K}[h,f] = n^{\frac{n}{2}} \int_{\mathbb{X}} K_{_I}(x,h)\,\mathrm{d}x + n^{\frac{n}{2}} \int_{\mathbb{Y}} K_{_I}(y,f)\,\mathrm{d}y \tag{19}$$

Minimizing this energy is a subject which hopefully will develop into a very coherent analogue of Teichmüller theory in higher dimensions.

For the applied aspect of our work we relied heavily on several reports of lectures and private communications [8]. Our list of references makes no pretense at being complete. Some references appear in the bibliography but are not specifically mentioned in this text.

2 Analytical Prerequisites

Apart from a few new symbols used in this text most of them are abundantly clear from the context.

2.1 List of symbols

$A \preccurlyeq B$	is merely an abbreviation for the inequality $A \leqslant \lambda \cdot B$ in which $\lambda > 0$, called the *implied constant*, plays no essential role. The quantities of interest to us are $A \geqslant 0$ and $B \geqslant 0$. The implied constant changes from line to line and can be easily identified from the context. Most frequently λ depends on the dimension only.
$\mathbb{R}^n_\circ$	is the Euclidean n-space with the origin removed, so-called *punctured space*, $\mathbb{R}^n_\circ = \{ x = (x_1, ..., x_n)\ ;\ \lvert x\rvert^2 = x_1^2 + \cdots + x_n^2 \ > 0\}$.
$\Vert A \Vert$	the Hilbert Schmidt norm, $\Vert A \Vert^2 = \mathrm{Tr}\,(A^* A)$.
$\mathbb{S}^{n-1}$	the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ of points $\omega = (x_1, ..., x_n)$ with $x_1^2 + ... x_n^2 = 1$.
$\mathrm{d}\omega$	the $(n-1)$-area element on $\mathbb{S}^{n-1}$, also identified with the restriction to $\mathbb{S}^{n-1}$ of the $(n-1)$ form

$$\Xi \overset{\text{def}}{=\!=} \sum_{i=1}^{n} (-1)^i \ \frac{x_i \,\mathrm{d}x_1 \wedge \cdots \widehat{\mathrm{d}x_i} \cdots \wedge \mathrm{d}x_n}{\lvert x\rvert^n} \qquad \text{on } \mathbb{R}^n_\circ \tag{20}$$

(r,ω)	polar coordinates in $\mathbb{R}^n_\circ$, where $r = \lvert x\rvert$ is referred to as the radial distance and $\omega = \frac{x}{\lvert x\rvert}$ as the spherical coordinate of x. Obviously $x = r\omega$ and $\mathrm{d}x = r^{n-1}\mathrm{d}r\,\mathrm{d}\omega$.

$\omega = (\theta, \mathfrak{s})$ *meridian coordinates* in the unit sphere, $0 \leqslant \theta \leqslant \pi$ and $\mathfrak{s} \in \mathbb{S}^{n-2}$. A position of a point $\omega \in \mathbb{S}^{n-1}$ is determined by its latitude and longitude. These are convenient coordinates for describing deformations of the unit sphere. We refer to the points $\omega_+ = (0, ..., 0, 1)$ and $\omega_- = (0, ..., 0, -1)$ as north and south poles, respectively. The unit sphere with poles removed will be denoted by $\mathbb{S}^{n-1}_{\pm}$. The equatorial sphere $\mathbb{S}^{n-2} \subset \mathbb{S}^{n-1}$ is given by the equations $x_1^2 + ... + x_{n-1}^2 = 1$, thus $x_n = 0$. To every point $\omega \in \mathbb{S}^{n-1}_{\pm}$ there correspond meridian coordinates

$$(\theta, \mathfrak{s}) \in (0, \pi) \times \mathbb{S}^{n-2} \sim \mathbb{S}^{n-1}_{\pm} \tag{21}$$

Here $\mathfrak{s}$ lies in the equatorial sphere while θ is the distance to the north pole measured in degrees. The rectangular coordinates of $\omega = (x_1, x_2, ..., x_n)$ are uniquely recovered from $\mathfrak{s}$ and θ by the rule

$$\omega = (\cos\theta\,,\, \mathfrak{s} \cdot \sin\theta)\,, \qquad 0 < \theta < \pi \tag{22}$$

The $(n-1)$-surface area of $\mathbb{S}^{n-1}$ is expressed in terms of the meridian angle as

$$\mathrm{d}\omega \;=\; (\sin\theta)^{n-2}\; \mathrm{d}\theta\, \mathrm{d}\mathfrak{s}\;, \qquad \int_{\mathbb{S}^{n-1}} \mathrm{d}\omega \;=\; \omega_{n-1} \tag{23}$$

where $\mathrm{d}\mathfrak{s}$ stands for the $(n-2)$-surface area of the equatorial sphere. Therefore,

$$\int_0^\pi \sin^{n-2}\theta \; \mathrm{d}\theta = \frac{\omega_{n-1}}{\omega_{n-2}}$$

u_r radial (or normal) derivative of u, identified with the n-form

$$u_r(x)\,\mathrm{d}x = \langle \omega, \nabla u \rangle\, \mathrm{d}x = \mathrm{d}u \wedge \star\, \mathrm{d}r \tag{24}$$

u_ω spherical (or tangential) derivative, identified with a 2-form

$$u_\omega = \mathrm{d}u \wedge \mathrm{d}r \;=\; \sum_{1 \leqslant i < j \leqslant n} \left(\frac{x_i\, \partial u}{\partial x_j} - \frac{x_j\, \partial u}{\partial x_i} \right) \frac{\mathrm{d}x_i \wedge \mathrm{d}x_j}{2\,|x|} \tag{25}$$

$\fint_{\mathbb{E}} u$ the integral average of u over a measurable set $\mathbb{E}$, usually over a ball $\mathbb{B}$, or a sphere $\mathbb{S}$.

2.2 Sobolev imbedding on spheres

Let u be a function on a sphere $\mathbb{S} = \mathbb{S}^{n-1}(a, r)$ in the Sobolev space $\mathscr{W}^{1,p}(\mathbb{S})$, $p > n-1$. Then u is continuous, and we have

$$\operatorname*{osc}_{\mathbb{S}}[u] \overset{\text{def}}{=\!=} \max_{x,y \in \mathbb{S}} |u(x) - u(y)\,| \;\leqslant A_p(n)\, r \left(\fint_{\mathbb{S}} |\nabla u(x)\,|^p \; \mathrm{d}x \right)^{\frac{1}{p}} \tag{26}$$

The smallest constant $A_p(n)$ is given by

$$\pi \leqslant A_p(n) \;=\; \left(\frac{\omega_{n-1}}{\omega_{n-2}}\right)^{\frac{1}{p}} \left[\int_0^\pi \frac{\mathrm{d}\theta}{\sin^{\frac{n-2}{p-1}}\,\theta}\right]^{\frac{p-1}{p}} \;\leqslant\; \frac{\pi(p-1)}{p-n+1} \tag{27}$$

see [39] and [25].

2.3 Oscillations on balls

Proposition 2.1. (OSCILLATION ON BALLS) *Every continuous injective map* $h \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$ *satisfies the inequality*

$$\left[\operatorname*{osc}_B h\right]^n \leqslant \frac{C_n}{\log \lambda} \int_{\lambda B} \|\, Dh(x)\, \|^{\,n}\, \mathrm{d}x \tag{28}$$

for all concentric balls $B \subsetneq \lambda B \subset \mathbb{X}$.

Choose and fix two points $a, b \in B(x_\circ, r) \subsetneq B(x_\circ, \lambda r) \subset \mathbb{X}$, where $x_\circ = \frac{1}{2}(a+b)$ and $r = \frac{1}{2}|a-b|$. For almost every $t \in (r, \lambda r)$ the map h has a trace along the sphere $\mathbb{S}_t = \{x\,;\;\; |x - x_\circ| = t\}$, which belongs to the Sobolev space $\mathscr{W}^{1,n}(\mathbb{S}_t, \mathbb{R}^m)$. By Sobolev inequality on spheres, we have

$$\left[\operatorname*{osc}_{\mathbb{S}_t} h\right]^n \preccurlyeq t \int_{\mathbb{S}_t} \|\, Dh\, \|^{\,n} \tag{29}$$

For every such t the unbounded component of $\mathbb{R}^m \setminus h(\mathbb{S}_t)$ neither contains $h(a)$ nor $h(b)$. This means that the straight line through $h(a)$ and $h(b)$ intersects $h(\mathbb{S}_t)$ at two points, say p and q, so that $h(a)$ and $h(b)$ lie between p and q. In particular,

$$|h(a) - h(b)| \leqslant |p - q| \leqslant \operatorname*{osc}_{\mathbb{S}_t} h \tag{30}$$

We combine this inequality with (29) to obatin

$$|h(a) - h(b)|^n \int_r^{\lambda r} \frac{\mathrm{d}t}{t} \preccurlyeq \int_r^{\lambda r} \left(\int_{\mathbb{S}_t} \|\, Dh\, \|^{\,n}\right) \mathrm{d}t \tag{31}$$

Hence

$$|h(a) - h(b)|^n \preccurlyeq \frac{1}{\log \lambda} \int_{\lambda B} \|\, Dh\, \|^{\,n}\,, \quad \lambda B = \{x\,;\;\; |x - x_\circ| \leqslant \lambda r\} \tag{32}$$

Slightly, more refined arguments actually give a uniform continuity estimate.

Lemma 2.1. (MODULUS OF CONTINUITY) *Let* $h \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$ *be a continuous injection. Then for every concentric balls* $\mathbb{B} \subset 2\mathbb{B} \subset \mathbb{X}$*, we have*

$$|h(x_1) - h(x_2)|^n \leqslant \frac{C_n \int_{2\mathbb{B}} \|\, Dh(x)\, \|^{\,n}\, \mathrm{d}x}{\log\left(e + \frac{\operatorname{diam} \mathbb{B}}{|x_1 - x_2|}\right)}\,, \qquad \textit{for} \quad x_1, x_2 \in \mathbb{B} \tag{33}$$

2.3.1 Oscillations on spheres

One useful consequence of the estimate (28) is the oscillation inequality on spheres in $\mathbb{R}^n$. Let $\mathbb{S}_\rho = \mathbb{S}^{\ell-1}(a,\rho) \subset \mathbb{R}^n$ be an $(\ell-1)$-dimensional sphere of radius ρ, $2 \leqslant \ell \leqslant n$. We consider the torus

$$\mathbb{T}_\rho = \{x \in \mathbb{R}^n\, ; \quad \operatorname{dist}(x, \mathbb{S}_\rho) < \rho\} \subset \mathbb{R}^n \tag{34}$$

Lemma 2.2. *Suppose $h \in \mathscr{W}^{1,n}(\mathbb{T}_\rho, \mathbb{R}^n)$ is a continuous injection. Then*

$$\left[\operatorname*{osc}_{S_\rho} h\right]^n \preccurlyeq \int_{\mathbb{T}_\rho} \|\, Dh \,\|^n \tag{35}$$

Proof. It is geometrically clear that every two points $x_1, x_2 \in \mathbb{S}_\rho$ can be connected by two closed balls $\overline{B_1}$ and $\overline{B_2}$ of radius ρ; that is, $x_1, x_2 \in \overline{B_1} \cup \overline{B_2}$, $\overline{B_1} \cap \overline{B_2} \neq \emptyset$ and $\sqrt{2}B_1 \cup \sqrt{2}B_2 \subset \mathbb{T}_\rho$. For this it is clearly sufficient to argue in dimension 2. Using (28), we conclude with the inequality

$$\begin{aligned} |h(x_1) - h(x_2)|^n \quad &\leqslant \quad \left[\operatorname*{osc}_{B_1} h + \operatorname*{osc}_{B_2} h\right]^n \\ &\preccurlyeq \quad \int_{\sqrt{2}B_1} \|\, Dh \,\|^n + \int_{\sqrt{2}B_2} \|\, Dh \,\|^n \preccurlyeq \int_{\mathbb{T}_\rho} \|\, Dh \,\|^n \end{aligned}$$

as desired.

Remark 2.1. Note that the oscillations $\operatorname*{osc}_{S_\rho} h$ are uniformly small as ρ goes to zero, because of equiintegrability of $\|\, Dh \,\|^n$ over the tori $\mathbb{T}_\rho$.

2.4 Quasiconformal change of variables

In what follows it will involve no loss of generality in assuming that the quasiconformally k-flat cracks are actually subsets of $\mathbb{R}^k$. This is a simple consequence of the following

Lemma 2.3. *Suppose $h : \mathbb{D} \to \mathbb{R}^n$ has finite conformal energy and $g : \mathbb{G} \xrightarrow{onto} \mathbb{D}$ is K-quasiconformal. Then the energy of $f = h \circ g : \mathbb{G} \to \mathbb{R}^n$ is uniformly controlled by that of h*

$$C_K^{-1}(n) \int_{\mathbb{D}} \|\, Dh(x) \,\|^n \, \mathrm{d}x \leqslant \int_{\mathbb{G}} \|\, Df(z) \,\|^n \, \mathrm{d}z \leqslant C_K(n) \int_{\mathbb{D}} \|\, Dh(x) \,\|^n \, \mathrm{d}x \tag{36}$$

Proof. It is legitimate to apply the chain rule to quasiconformal mappings. Accordingly, $Df(z) = Dh(x)Dg(z)$, where $x = g(z)$. Hence

$$\|\, Df(z) \,\|^n \leqslant \|\, Dh(x) \,\|^n \,\|\, Dg(z) \,\|^n \leqslant n^{\frac{n}{2}}\, K \,\|\, Dh(x) \,\|^n J(z,g) \tag{37}$$

by the distortion inequality $\| Dg(z) \|^n \leqslant n^{\frac{n}{2}} K\, J(z,g)$. Integration by substitution yields the desired upper bound of the energy of f. Likewise, this estimate applied to the inverse of g, yields the lower bound.

3 Topological Prerequisites

For the convenience of the reader we collect here some results concerning connectedness, dimension and separation of $\mathbb{R}^n$, see [1, 23, 35, 48, 49]. Perhaps the less known are the Joshi's Antipodal Theorem and Lafont's Strong Separation Theorem.

THEOREM 3.1. (Unicoherence of $\mathbb{R}^n$) *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be domains in* $\mathbb{R}^n$*, open connected subsets, such that* $\mathbb{A} \cup \mathbb{B} = \mathbb{R}^n$*. Then* $\mathbb{A} \cap \mathbb{B}$ *is also a domain.*

THEOREM 3.2. *A set (not necessarily closed)* $\mathfrak{X} \subset \mathbb{R}^n$ *has no interior points if and only if* $\dim \mathfrak{X} \leqslant n-1$.

THEOREM 3.3. *No closed set* $\mathfrak{X} \subset \mathbb{R}^n$ *with* $\dim \mathfrak{X} \leqslant n-2$ *separates a domain* $\mathbb{U} \subset \mathbb{R}^n$*. That is,* $\mathbb{U} \setminus \mathfrak{X}$ *is connected.*

The classical Jordan Curve Theorem, generalized to higher dimensions by Brouwer [17] asserts that

THEOREM 3.4. (Jordan-Brouwer Separation Theorem) *A subset* $\mathcal{S} \subset \mathbb{R}^n$ *which is homeomorphic to* $\mathbb{S}^{n-1}$ *disconnects* $\mathbb{R}^n$ *into two components. That is,* $\mathbb{R}^n \setminus \mathcal{S}$ *consists of two disjoint connected open sets* $\mathbb{U}$ *and* $\mathbb{V}$ *whose common boundary is* $\mathcal{S}$.

An intrinsic characterization of disconnecting compact sets in $\mathbb{R}^n$ is due to K. Borsuk, see [12]. But we shall not enter into this theory here as the need will not arise. However, the following corollary will be useful.

THEOREM 3.5. *If a compact set* $\mathfrak{X} \subset \mathbb{R}^n$ *disconnects* $\mathbb{R}^n$ *then so does its homeomorphic image in* $\mathbb{R}^n$*. The number of components in* $\mathbb{R}^n \setminus \mathfrak{X}$ *is a topological invariance. Let us also note an obvious fact that a connected open set* $\Omega \subset \mathbb{R}^n \setminus \mathfrak{X}$ *is a component of* $\mathbb{R}^n \setminus \mathfrak{X}$ *if and only if* $\partial\Omega \subset \mathfrak{X}$.

Quite recently Theorem 3.4 has been generalized to mappings that are not necessarily injective on the entire sphere $\mathbb{S}^{n-1}$, see J-F Lafont [50]. We shall not need Lafont's result in its full generality here. Instead, we extract the following from Theorem 1.3 in [50].

THEOREM 3.6. *Let* $\mathcal{S} \subset \mathbb{R}^n$ *be homeomorphic to* $\mathbb{S}^{n-1}$ *and* $\mathfrak{X} \subsetneq \mathcal{S}$ *a proper compact subset of* $\mathcal{S}$. *Suppose we are given a continuous map* $h : \mathcal{S} \to \mathbb{R}^n$, *which is injective on* $\tilde{\mathcal{S}} = \mathcal{S} \setminus \mathfrak{X}$ *and* $h(\tilde{\mathcal{S}}) \cap h(\mathfrak{X}) = \emptyset$. *Then*

- *The set* $\Gamma = h(\mathcal{S})$ *separates the space* $\mathbb{R}^n$.
- *To every point* $p \in \tilde{\Gamma} = h(\tilde{\mathcal{S}})$ *there correspond precisely two connected components* $\mathbb{U}, \mathbb{V} \subset \mathbb{R}^n \setminus \Gamma$ *which contain* p *in their closure.*

As stated in Theorem 3.4 the homeomorphic image of $\mathbb{S}^{n-1}$ separates $\mathbb{R}^n$ into two open sets, the inside and the outside. In the planar case these two sets are homeomorphic to the inside and outside of a standard circle. Such a theorem is only valid in two dimensions. In three dimensions there are counterexamples such as Alexander's horned sphere, see [11]. There exist, however, a higher-dimensional generalization due to M. Brown, [18] and independently to B. Mazur, [58] with M. Morse [61]. Their result is better known to as the Schönflies theorem.

THEOREM 3.7. (SCHÖNFLIES THEOREM) *Let* $\mathbb{S}^{n-1}_\epsilon = \{x \in \mathbb{R}^n\,;\quad 1-\epsilon < |x| < 1+\epsilon\}$ *and* $f : \mathbb{S}^{n-1}_\epsilon \to \mathbb{R}^n$ *be a continuous injection. Then for every* $0 < \delta < \epsilon$ *there is a homeomorphism* $F : \mathbb{R}^n \xrightarrow{onto} \mathbb{R}^n$ *such that* $F = f$ *on* $\mathbb{S}^{n-1}_\delta$.

We shall make uppeal to a generalization of the celebrated antipodal theorem of Borsuk and Ulam [13]. The interested reader may consult [75], [62], for various generalizations. The nice variant, and very useful for us, belongs to K. D. Joshi [42].

THEOREM 3.8. (JOSHI'S ANTIPODAL THEOREM) *Let* Ω *be a bounded open subset of* $\mathbb{R}^{k+1}$ *containing the origin and let* $\Phi : \partial\Omega \to \mathbb{R}^k$ *be a continuous map. Then there exist so-called antipodal points* $\xi_1, \xi_2 \in \partial\Omega$ *at which* Φ *assumes the same value. Precisely, we have*

$$\frac{\xi_1}{|\xi_1|} + \frac{\xi_2}{|\xi_2|} = 0\,, \qquad \textit{and} \quad \Phi(\xi_1) \;=\; \Phi(\xi_2) \tag{38}$$

4 Cracks and Internal Fractures

Let $\mathbb{X}$ be a bounded domain in $\mathbb{R}^n$. A *crack* in $\mathbb{X}$ is a relatively closed subset $\mathfrak{X} \subset \mathbb{X}$ having no interior (thus $\dim \mathfrak{X} \leqslant n-1$) and such that $\mathbb{X} \setminus \mathfrak{X}$ is connected. Note that for cracks of dimension $\dim \mathfrak{X} \leqslant n-2$, the connectedness of $\mathbb{X} \setminus \mathfrak{X}$ is automatic by Theorem 3.3. A crack which is connected and compactly contained in $\mathbb{X}$ will be called *internal fracture*. Here and throughout we use the notation

$$\tilde{\mathbb{A}} = \mathbb{A} \setminus \mathfrak{X} \quad \text{for every set } \mathbb{A} \subset \mathbb{R}^n \tag{39}$$

4.1 Cluster set of a homeomorphism

We shall recall some terminology useful in the discussion of the behavior of homeomorphisms near the boundary, in particular near cracks and fractures in $\mathbb{X}$. Let $h : \mathbb{D} \to \mathbb{R}^n$ be a bounded homeomorphism defined in a bounded domain $\mathbb{D} \subset \mathbb{R}^n$. Given a point $a \in \overline{\mathbb{D}}$, we consider all possible sequential limits, $\lim h(x_i)$, as $x_i \to a$ and $x_i \in \mathbb{D}$. The cluster set of h at a, denoted by $h\{a\}$, consists of all such sequential limits. In particular, $h\{a\} = \{h(a)\}$ for points where h is continuous. In the same vain we define the cluster set of h over any subset of $\overline{\mathbb{D}}$.

$$h\{\mathbb{A}\} = \bigcup_{a \in \mathbb{A}} h\{a\} \,, \qquad \text{for any } \mathbb{A} \subset \overline{\mathbb{D}}$$

Obviously, $h\{\}$ takes closed subsets of $\overline{\mathbb{D}}$ into closed subsets in $\mathbb{R}^n$. Since $h : \mathbb{D} \to \mathbb{R}^n$ is a homeomorphism, we find that,

$$\partial h(\mathbb{D}) = h\{\partial \mathbb{D}\} \; ; \quad \text{in particular, } h\{\partial \mathbb{D}\} \cap h(\mathbb{D}) = \emptyset \tag{40}$$

The proof of these facts is an easy exercise in the point set topology, which we leave to the reader. Perhaps less obvious is the connectedness of the cluster set of a continuum of dimension $\leqslant n - 2$.

Lemma 4.1. *Let $\mathbb{X}$ be a bounded domain in $\mathbb{R}^n$, $\mathfrak{X}$ a crack in $\mathbb{X}$ of dimension $\dim \mathfrak{X} \leqslant n - 2$ and $h : \tilde{\mathbb{X}} \to \mathbb{R}^n$ a bounded injective map, $\tilde{\mathbb{X}} = \mathbb{X} \setminus \mathfrak{X}$. Then $h\{\}$ takes continua in $\mathfrak{X}$ into continua. Moreover,*

$$h(\tilde{\mathbb{X}}) \cap h\{\mathfrak{X}\} = \emptyset \tag{41}$$

Proof. Let $\mathbb{A}$ be a continuum in $\mathfrak{X}$. Consider connected level sets of $\mathbb{A}$, defined by $\mathbb{A}_k = \{x \in \mathbb{R}^n;\ \operatorname{dist}(x, \mathbb{A}) < \frac{1}{k}\} \subset \mathbb{X}$, for sufficiently large integers k. By Theorem 3.3 the sets $\tilde{\mathbb{A}}_k = \mathbb{A}_k \setminus \mathfrak{X}$ are connected, so do the images $h(\tilde{\mathbb{A}}_k)$ and their closure. To complete the proof we need only observe that $h\{\mathbb{A}\} = \bigcap \overline{h(\tilde{\mathbb{A}}_k)}$, which is connected.

Remark 4.1. Lemma 4.1 fails if $\dim \mathfrak{X} = n - 1$, see Example 1.1.

5 Proof of Proposition 1.2

Suppose that $x_\circ = 0$. We look at the images $\Gamma_\epsilon = h(\mathbb{S}_\epsilon)$ of the concentric spheres $\mathbb{S}_\epsilon = \{x\,;\ |x-x_\circ| = \epsilon\}$, for $3\epsilon < \operatorname{dist}(x_\circ, \partial\Omega) \overset{\text{def}}{=\!=} r$. These are topological $(n-1)$-spheres imbedded in $\mathbb{R}^n$. By Jordan-Brouwer Separation Theorem 3.4, each Γ_ϵ separates $\mathbb{R}^n$ into two components of which Γ_ϵ is their common boundary. We denote by B_ϵ the bounded component of $\mathbb{R}^n \setminus \Gamma_\epsilon$. The family $\{\Gamma_\epsilon\}_{0<\epsilon<r}$ is monotone, meaning that there are two possibilities: either $B_\epsilon \subset B_\rho$ for all $0 < \epsilon < \rho < r$, or $B_\epsilon \supset B_\rho$ for all $0 < \epsilon < \rho < r$. This latter possibility is ruled out by Lemma 2.2. Indeed, if not, then letting $\epsilon \to 0$ we would obtain a contradiction $0 < \dim B_\rho \leqslant \operatorname{diam} B_\epsilon = \operatorname*{osc}_{\mathbb{S}_\epsilon} h \to 0$. Thus, we have a decreasing family of compact sets $\overline{B_\epsilon}$. Its intersection consists of a single point,

$$\bigcap_{0<\epsilon<r} \overline{B_\epsilon} = \{y_\circ\}$$

It is plain that $y_\circ = \lim_{x \to x_\circ} h(x)$, so we may extend h to $\mathbb{X}$ by letting $h(x_\circ) = y_\circ$. The injectivity of h is routine and left to the reader.

6 Proof of Theorem 1.1

The proof of Theorem 1.1 is lengthy and rather complex. It consists of 4 steps, through the subsections 6.1-6.4.

6.1 Separation Lemma

Throughout this subsection $\mathbb{X}$ will be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, and $\mathfrak{X}$ a relatively closed subset of $\mathbb{X}$ with $\dim \mathfrak{X} = k \leqslant n-2$. We recall that $\mathfrak{X}$ does not separate any open connected set $\Omega \subset \mathbb{R}^n$; that is, $\Omega \setminus \mathfrak{X}$ is a domain (still a solid body). We recall the following notation: whenever the crack $\mathfrak{X}$ is removed from a set $\mathbb{A} \subset \mathbb{R}^n$ the remaining part is denoted by

$$\tilde{\mathbb{A}} = \mathbb{A} \setminus \mathfrak{X}\,, \qquad \text{thus the set} \quad \tilde{\mathbb{X}} = \mathbb{X} \setminus \mathfrak{X} \quad \text{is a domain.}$$

Suppose we are given a set $\mathcal{S} \subset \mathbb{X}$ homeomorphic to $\mathbb{S}^{n-1}$. We denote by $\mathbb{E}$ the bounded component of $\mathbb{R}^n \setminus \mathcal{S}$ and assume that it lies in $\mathbb{X}$. In particular, $\mathbb{X}$ and the unbounded component of $\mathbb{R}^n \setminus \mathcal{S}$ cover the entire space $\mathbb{R}^n$. Because of unicoherence of the Euclidean space, see Theorem 3.1, we find that the open set $\mathbb{F} = \mathbb{X} \setminus \overline{\mathbb{E}}$ is also connected. Therefore, we have a disjoint decomposition $\mathbb{X} = \mathbb{E} \cup \mathcal{S} \cup \mathbb{F}$, where $\mathbb{E}$ and $\mathbb{F}$ are subdomains in $\mathbb{X}$ having $\mathcal{S}$ as their common boundary. Since $\dim \mathfrak{X} \leqslant n-2$, removing $\mathfrak{X}$ we obtain a disjoint decomposition of $\tilde{\mathbb{X}} = \mathbb{X} \setminus \mathfrak{X}$,

$$\tilde{\mathbb{X}} = \tilde{\mathbb{E}} \cup \tilde{\mathcal{S}} \cup \tilde{\mathbb{F}} \, , \qquad \tilde{\mathcal{S}} \subset \partial \tilde{\mathbb{E}} \cap \partial \tilde{\mathbb{F}}$$

where $\tilde{\mathbb{E}}$ and $\tilde{\mathbb{F}}$ are disjoint subdomains of $\tilde{\mathbb{X}}$, each of which contains $\tilde{\mathcal{S}}$ in its boundary. In general, $\tilde{\mathcal{S}}$ need not be connected. The above arguments may further be extended to show that in fact $\mathcal{S} \subset \partial \tilde{\mathbb{E}} \cap \partial \tilde{\mathbb{F}}$. For this we only need to notice that $\tilde{\mathcal{S}}$ is dense in the space $\mathcal{S}$. This is true because we have removed from $\mathcal{S}$ a set of dimension $k < n-1 = \dim \mathcal{S}$, see Theorem 3.2.

We will be given a homeomorphism $h : \tilde{\mathbb{X}} \to \mathbb{R}^n$ which extends continuously to $\mathcal{S}$. The same letter h will be used to denote this continuous extension, $h : \mathcal{S} \cup \tilde{\mathbb{X}} \to \mathbb{R}^n$. Note that $h(\tilde{\mathcal{S}})$ lies in the common boundary of $h(\tilde{\mathbb{E}})$ and $h(\tilde{\mathbb{F}})$. Moreover, the set

$$h(\tilde{\mathbb{E}}) \cup h(\tilde{\mathcal{S}}) \cup h(\tilde{\mathbb{F}}) \; = \; h(\tilde{\mathbb{E}} \cup \tilde{\mathcal{S}} \cup \tilde{\mathbb{F}}) \; = \; h(\tilde{\mathbb{X}}) \; , \quad \text{is a domain.}$$

We shall now reformulate Theorem 3.6 in order to use it for the mapping $h : \mathcal{S} \cup \tilde{\mathbb{X}} \to \mathbb{R}^n$.

Lemma 6.1. *The set* $\Gamma = h(\mathcal{S})$ *separates* $\mathbb{R}^n$ *. There are precisely two components* $\mathbb{U}, \mathbb{V}$ *of* $\mathbb{R}^n \setminus \Gamma$ *such that* $h(\tilde{\mathbb{E}}) \subset \mathbb{U}$ *and* $h(\tilde{\mathbb{F}}) \subset \mathbb{V}$ *. These components enjoy the following properties:*

- *The boundary of each component* $\mathbb{U}$ *and* $\mathbb{V}$ *contains* $\tilde{\Gamma} = h(\tilde{\mathcal{S}})$*. The closure of no other component of* $\mathbb{R}^n \setminus \Gamma$ *intersects* $\tilde{\Gamma}$*.*
- *Gluing the components: The set* $\Omega = \mathbb{U} \cup \tilde{\Gamma} \cup \mathbb{V}$ *is a domain.*

Proof. First we observe that each of the connected sets $h(\tilde{\mathbb{E}})$ and $h(\tilde{\mathbb{F}})$ lies in $\mathbb{R}^n \setminus \Gamma$. Indeed, we have $h(\tilde{\mathbb{E}}) \cap \Gamma = h(\tilde{\mathbb{E}}) \cap [\tilde{\Gamma} \cup h(\mathcal{S} \cap \mathfrak{X})] \subset h(\tilde{\mathbb{X}}) \cap h\{\mathfrak{X}\} = \emptyset$, by (41). The same holds for $h(\tilde{\mathbb{E}})$. Similarly, $h(\tilde{\mathcal{S}}) \cap h(\mathfrak{X}) \subset h(\tilde{\mathbb{X}}) \cap h\{\mathfrak{X}\} = \emptyset$. From the first part of Theorem 3.6, we infer that the set $\Gamma = h(\mathcal{S})$ separates $\mathbb{R}^n$. Each of the two disjoint connected sets $h(\tilde{\mathbb{E}})$, $h(\tilde{\mathbb{F}}) \subset \mathbb{R}^n \setminus \Gamma$ must lie entirely in one and only one component of $\mathbb{R}^n \setminus h(\mathcal{S})$. Suppose that $h(\tilde{\mathbb{E}}) \subset \mathbb{U}$ and $h(\tilde{\mathbb{F}}) \subset \mathbb{V}$. We emphasize that the components $\mathbb{U}, \mathbb{V} \subset \mathbb{R}^n \setminus \Gamma$ are uniquely determined by the sets $\tilde{\mathbb{E}}$ and $\tilde{\mathbb{F}}$. We aim to show that $\mathbb{U}$ and $\mathbb{V}$ are different. To this effect the second part of Theorem 3.6 comes into play. It tells us that for every point $p \in \tilde{\Gamma} = h(\tilde{\mathcal{S}})$ there are precisely two components of $\mathbb{R}^n \setminus \Gamma$ which contain p in their closure. Let us denote them by $\mathbb{U}(p)$ and $\mathbb{V}(p)$. On the other hand, the sets $h(\tilde{\mathbb{E}})$, $h(\tilde{\mathbb{F}}) \subset \mathbb{R}^n \setminus \Gamma$ contain p in their closure, so do the components $\mathbb{U}$ and $\mathbb{V}$. This shows that each of the sets $\mathbb{U}$ and $\mathbb{V}$ coincides with either $\mathbb{U}(p)$ or $\mathbb{V}(p)$. Now suppose, to the contrary, that $\mathbb{U} = \mathbb{V}$. Thus both sets $h(\tilde{\mathbb{E}})$ and $h(\tilde{\mathbb{F}})$ lie in the same component of $\mathbb{R}^n \setminus \Gamma$. Suppose $\mathbb{U}, \mathbb{V} \subset \mathbb{U}(p)$. Consequently, the domain $h(\tilde{\mathbb{X}}) = h(\tilde{\mathbb{E}}) \cup h(\tilde{\mathcal{S}}) \cup h(\tilde{\mathbb{F}})$ lies in the closure of $\mathbb{U}(p)$. Here we observe that p is the interior point of $h(\tilde{\mathbb{X}})$. This means that p is also the interior point of $\mathbb{U}(p)$ and as such cannot belong to the closure of $\mathbb{V}(p)$. In conclusion, $\mathbb{U}$ and $\mathbb{V}$ are different components of $\mathbb{R}^n \setminus \Gamma$. Let us stress here that $\mathbb{U}$ and $\mathbb{V}$ coincide with $\mathbb{U}(p)$ and $\mathbb{V}(p)$, respectively, for every p in $\tilde{\Gamma} = h(\tilde{\mathcal{S}})$. Finally, the set $\Omega = \mathbb{U} \cup \tilde{\Gamma} \cup \mathbb{V} = \mathbb{U} \cup [h(\tilde{\mathbb{E}}) \cup h(\tilde{\mathcal{S}}) \cup h(\tilde{\mathbb{F}})] \cup \mathbb{V} = \mathbb{U} \cup h(\tilde{\mathbb{X}}) \cup \mathbb{V}$ is a domain. The proof of the lemma is complete.

We refer the reader to Remark 9.1 as an illustration to this lemma.

6.2 A homotopy path from ellipsoids to the crack

We shall now proceed to a more concrete realization of Lemma 6.1. Choose and fix an integer $1 \leqslant k \leqslant n-2$. We view $\mathbb{R}^n$ as the Cartesian product, $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, in which the coordinates of a point $x \in \mathbb{R}^n$ are split into two components as: $x = (w, z)$, where $w = (x_1, ..., x_k)$ and $z = (x_{k+1}, ..., x_n)$. Accordingly, the unit ball and its boundary are given by

$$\mathbb{B} = \left\{(w,z) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \; ; \quad |w|^2 + |z|^2 < 1\right\} \tag{42}$$

$$\mathbb{S}^{n-1} = \left\{(w,z) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \; ; \quad |w|^2 + |z|^2 = 1\right\} \tag{43}$$

The subspace $\mathfrak{X} = \mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$ will be viewed as a crack in $\mathbb{R}^n$. We shall work with one parameter family of ellipsoids in $\mathbb{R}^n$, which connects the unit sphere $\mathbb{S}^{n-1}$ with the crack $\mathfrak{X} \cap \mathbb{B}$. Precisely, for $0 < \lambda \leqslant 1$, we consider the ellipsoid

$$\mathcal{S}_\lambda = \left\{(w,z) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \; ; \; \lambda^2|w|^2 + |z|^2 = \lambda^2\right\}, \text{ thus } \mathcal{S}_1 = \mathbb{S}^{n-1}$$

and its interior

$$\mathbf{B}_\lambda = \left\{(w,z) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \; ; \quad \lambda^2|w|^2 + |z|^2 < \lambda^2\right\}, \text{ thus } \mathbf{B}_1 = \mathbb{B}$$

Then we remove the crack to denote

$$\tilde{\mathbf{B}}_\lambda = \mathbf{B}_\lambda \setminus \mathfrak{X} = \left\{(w,z) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \; ; \quad 0 < |z|^2 < \lambda^2 - \lambda^2|w|^2 \right\}$$

It is important to emphasize that the sphere $\mathbb{S}^{k-1} = \{(w,0);\ |w| = 1\} = \mathfrak{X} \cap \mathbb{S}^{n-1}$ is the common part of all ellipsoids $\mathcal{S}_\lambda$, $0 < \lambda \leqslant 1$, and no two ellipsoids in the family have other points in common. Continuing in this fashion, the remaining part will be denoted by

$$\tilde{\mathcal{S}}_\lambda = \mathcal{S}_\lambda \setminus \mathfrak{X} = \big\{(w,z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}\ ; \quad |z|^2 = \lambda^2 - \lambda^2 |w|^2 \ > \ 0\big\}\,.$$

This is an open connected subset of $\mathcal{S}_\lambda$. Finally, for $0 < \alpha < \beta$, we introduce the domains:

$$\mathbf{B}_\alpha^\beta = \mathbf{B}_\beta \setminus \overline{\mathbf{B}}_\alpha = \big\{(w,z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}\ ; \quad \alpha^2(1-|w|^2) < |z|^2 < \beta^2(1-|w|^2)\big\}$$

whose boundary consists of three disjoint parts

$$\partial \mathbf{B}_\alpha^\beta = \mathcal{S}_\beta \cup \mathcal{S}_\alpha = \tilde{\mathcal{S}}_\beta \cup \tilde{\mathcal{S}}_\alpha \cup \mathbb{S}^{k-1} \quad \text{- disjoint decomposition}$$

6.3 The image of the ellipsoids

Let $\mathbb{X} \subset \mathbb{R}^n$ be a bounded domain and assume that it contains the closed unit ball $\overline{\mathbb{B}}$. We will be given a homeomorphism $h : \mathbb{X} \setminus \mathfrak{X} \to \mathbb{R}^n$, $\mathfrak{X} = \mathbb{R}^k \times \{0\}$, which extends continuously to $\mathbb{S}^{n-1}$, again denoted by $h : \mathbb{S}^{n-1} \to \mathbb{R}^n$. It turns out that all ellipsoids in the family share this property, namely

$$h : \mathcal{S}_\lambda \to \mathbb{R}^n \quad \text{is continuous for all } 0 < \lambda \leqslant 1\ . \tag{44}$$

More generally, the inclusion $\mathcal{S}_\lambda \subset \overline{\mathbf{B}} \setminus \mathbf{B}_\varepsilon$ holds whenever $0 < \varepsilon < \lambda$ and we have

Lemma 6.2. *The map*

$$h : \overline{\mathbf{B}} \setminus \mathbf{B}_\varepsilon \ \to \mathbb{R}^n \quad \textit{is continuous for all } 0 < \varepsilon \leqslant 1\ . \tag{45}$$

Proof. This fact is due to the oscillation inequality in Proposition 2.1. We need to examine a sequence $x_i \in \overline{\mathbf{B}} \setminus \mathbf{B}_\varepsilon$ converging to a point $a \in \mathbb{S}^{k-1} = \mathfrak{X} \cap \mathbb{S}^{n-1}$, as this is the only nontrivial situation to be verified. Thus $|x_i| \to 1$. Let us project these points onto the unit sphere $\mathbb{S}^{n-1}$ to obtain a sequence $a_i = \frac{x_i}{|x_i|} \to a$. The closed ball $B_i = \{x;\ |x - a_i| \leqslant r_i = 1 - |x_i|\}$ contains both $a_i \in \mathbb{S}^{n-1}$ and $x_i \in \overline{\mathbf{B}} \setminus \mathbf{B}_\varepsilon$. For sufficiently large i we have $|x_i| > 1 - \varepsilon^3$ and, consequently, $B_i \subset 2B_i \subset \mathbb{X} \setminus \mathfrak{X}$. It is immediate from an elementary geometric computation that

$$\operatorname{dist}(a_i, \mathfrak{X}) \ \geqslant \ \sqrt{\frac{\varepsilon^2}{1-\varepsilon^2}\,\frac{1+|x_i|}{1-|x_i|}}\ r_i > 2r_i$$

Applying the oscillation inequality (28) we obtain

$$|\,h(x_i) - h(a)\,| \ \leqslant \ |h(x_i) - h(a_i)| + |\,h(a_i) - h(a)\,| \ \leqslant \ \operatorname*{osc}_{B_i}\ h + |\,h(a_i) - h(a)\,| \ \to \ 0$$

because $h(a_i) \to h(a)$ and $|B_i| \to 0$, when we approach the set $\mathbb{S}^{k-1}$. This ends the proof of (45).

In the next step we shall see how the above ellipsoids are deformed under the mapping h. Meanwhile let us introduce the notation for the images of the ellipsoids and record basic relations between them. For $0 < \varepsilon < \lambda \leqslant 1$, we set:

$$\Gamma_\lambda = h(\mathcal{S}_\lambda)\ ,\ \ \tilde{\Gamma}_\lambda = h(\tilde{\mathcal{S}}_\lambda)\ ,\ \mathbf{S} = h(\mathbb{S}^{k-1}) = h(\mathcal{S}_\lambda \cap \mathfrak{X}) \quad \text{and} \ \ \Delta^\lambda_\varepsilon = h(\mathbf{B}^\lambda_\varepsilon)$$

We have the following disjoint decompositions:

$$\Gamma_\lambda = \tilde{\Gamma}_\lambda \cup \mathbf{S}\ , \qquad \partial \Delta^\lambda_\epsilon = \tilde{\Gamma}_\lambda \cup \tilde{\Gamma}_\varepsilon \ \cup \mathbf{S} \tag{46}$$

That these decompositions are disjoint follows from (41).

Proposition 6.1. *Each* Γ_λ , $0 < \lambda \leqslant 1$ *, separates the space* $\mathbb{R}^n$ *and* $h(\tilde{\mathbf{B}}_\lambda)$ *lies in a bounded component of* $\mathbb{R}^n \setminus \Gamma_\lambda$ *.*

That Γ_λ separates $\mathbb{R}^n$ is immediate from Theorem 3.6, applied to the ellipsoid $\mathcal{S}_\lambda$. The seemingly easy second part of this Proposition turns out to be rather cumbersome. As a convincing argument, let us take a quick look at the homeomorphism of the punctured space $\mathbb{R}^n_\circ$ onto the complement of the unit ball, $h(x) = 2x\,(|x| + |x|^2)^{-1}$. Its gradient lies in the Marcinkiewicz weak space $\mathscr{L}^n_{\text{weak}}(\mathbb{R}^n)$, which places h in the class of nearly finite conformal energy. Moreover, $h = \text{id} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. Nevertheless, h takes the unit punctured ball outside the sphere. Of course a cavitation has occurred, and $h \notin \mathscr{W}^{1,n}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$.

6.3.1 Proof of Proposition 6.1

Let us commence with a contrary supposition to Proposition 6.1 state it as.
Supposition. *The set* $h(\tilde{\mathbf{B}}_\lambda)$ *lies in the unbounded component of* $\mathbb{R}^n \setminus \Gamma_\lambda$ *.*

This supposition will stand until we reach a contradiction at the end of this section. Thus some of the subsequent statements look strange, at least intuitively, because the supposition actually false.

We shall work with one-parameter family of ellipsoids $\mathcal{S}_\varepsilon$ and $\tilde{\mathcal{S}}_\varepsilon = \mathcal{S}_\varepsilon \setminus \mathfrak{X}$, for $0 < \varepsilon < \lambda \leqslant 1$, and their images $\Gamma_\varepsilon = h(\mathcal{S}_\varepsilon)$ and $\tilde{\Gamma}_\varepsilon = h(\tilde{\mathcal{S}}_\varepsilon)$. Lemma 6.1 yields:

Corollary 6.1. *The set* Γ_ε *separates* $\mathbb{R}^n$*, for every* $0 < \varepsilon \leqslant 1$ *. There are precisely two components* $\mathbb{U}_\varepsilon, \mathbb{V}_\varepsilon \subset \mathbb{R}^n \setminus \Gamma_\varepsilon$ *each of which contains* $\tilde{\Gamma}_\varepsilon$ *in its boundary. The closure of no other component of* $\mathbb{R}^n \setminus \Gamma_\varepsilon$ *intersects* $\tilde{\Gamma}_\varepsilon$ *. The set* $\mathbb{U}_\varepsilon \cup \tilde{\Gamma}_\varepsilon \cup \mathbb{V}_\varepsilon$ *is a domain.*

Under our Supposition, we have:

Lemma 6.3. *For* $0 < \varepsilon < \lambda \leqslant 1$ *the domain* $\Delta_\varepsilon^\lambda$ *lies in a bounded component of* $\mathbb{R}^n \setminus \Gamma_\varepsilon$ *.*

Proof. We look at the boundary of $\Delta_\varepsilon^\lambda = h(\mathbf{B}_\varepsilon^\lambda)$ and observe that

$$\begin{gathered}\partial\Delta_\varepsilon^\lambda = \tilde{\Gamma}_\lambda \cup \mathbf{S} \cup \tilde{\Gamma}_\varepsilon = \Gamma_\lambda \cup \tilde{\Gamma}_\varepsilon = \tilde{\Gamma}_\lambda \cup \Gamma_\varepsilon \\ \Delta_\varepsilon^\lambda \subset \mathbb{R}^n \setminus \Gamma_\lambda \text{ and } \quad \Delta_\varepsilon^\lambda \subset \mathbb{R}^n \setminus \Gamma_\varepsilon\end{gathered} \tag{47}$$

We then infer from Corollary 6.1 that the domain $\Delta_\varepsilon^\lambda$ must lie entirely in one of the components $\mathbb{U}_\varepsilon$, $\mathbb{V}_\varepsilon \subset \mathbb{R}^n \setminus \Gamma_\varepsilon$, because $\tilde{\Gamma}_\epsilon$ lies in its closure. Assume that $\Delta_\varepsilon^\lambda \subset \mathbb{U}_\varepsilon$. Now it remains to show that $\mathbb{U}_\varepsilon$ is a bounded component. Suppose otherwise is true. Then $\mathbb{V}_\varepsilon \subset \mathbb{R}^n \setminus \Gamma_\epsilon$ must be bounded, because there can be only one unbounded component. Concerning the boundary of $\mathbb{V}_\varepsilon$, we observe that $\partial\mathbb{V}_\varepsilon \subset \Gamma_\varepsilon = \tilde{\Gamma}_\varepsilon \cup \mathbf{S} \subset \tilde{\Gamma}_\varepsilon \cup \Gamma_\lambda$. We now glue two domains $\Delta_\varepsilon^\lambda$ and $\mathbb{V}_\varepsilon$ along $\tilde{\Gamma}_\epsilon$ to obtain

$$\mathcal{D} = \Delta_\varepsilon^\lambda \cup \tilde{\Gamma}_\varepsilon \cup \mathbb{V}_\varepsilon = \Delta_\varepsilon^\lambda \cap \Big[\mathbb{U}_\varepsilon \cup \tilde{\Gamma}_\varepsilon \cup \mathbb{V}_\varepsilon\Big] \subset \mathbb{R}^n \setminus \Gamma_\lambda$$

The set within the rectangular parentheses is a domain, by Corollary 6.1. Thus $\mathcal{D}$ is an open subset in $\mathbb{R}^n \setminus \Gamma_\lambda$. We shall now argue that $\mathcal{D}$ is connected. For this we express $\mathcal{D}$ as

$$\mathcal{D} = \Big(\Delta_\varepsilon^\lambda \cup \tilde{\Gamma}_\varepsilon\Big) \cup \Big(\tilde{\Gamma}_\varepsilon \cup \mathbb{V}_\varepsilon\Big)$$

Here, the first set in the round parentheses lies between $\Delta_\varepsilon^\lambda$ and its closure, so is connected. The second set lies between $\mathbb{V}_\varepsilon$ and its closure, so is connected too. The union of overlapping connected sets is connected.

Concerning the boundary of $\mathcal{D}$ we find that $\partial\mathcal{D} \subset \partial\,\Delta_\varepsilon^\lambda \cup \partial\mathbb{V}_\varepsilon \subset \Gamma_\lambda \cup \tilde{\Gamma}_\varepsilon$. But $\tilde{\Gamma}_\varepsilon$ lies in the domain $\mathcal{D}$, so $\partial\mathcal{D} \subset \Gamma_\lambda$. This means that $\mathcal{D}$ is a component of $\mathbb{R}^n \setminus \Gamma_\lambda$, see Theorem 3.5. Of course, this must be a bounded component. Finally, in view of the Supposition, we arrive at the contradiction because $\Delta_\varepsilon^\lambda \subset h(\mathbf{B}^\lambda \setminus \mathfrak{X})$ and $h(\mathbf{B}^\lambda \setminus \mathfrak{X})$ lies in the unbounded component of $\mathbb{R}^n \setminus \Gamma_\lambda$. This establishes Lemma 6.3.

In our next step we make some preparation for the use Joshi's Antipodal Theorem.

Lemma 6.4. (Topological homogeneity of domains) *For every pair* a, b *of points in a domain* $\mathbb{G} \subset \mathbb{R}^n$ *there exists a homeomorphism* $\Psi : \mathbb{R}^n \xrightarrow{onto} \mathbb{R}^n$ *which is the identity on* $\mathbb{R}^n \setminus \mathbb{G}$ *and takes* a *into* b*,* $\Psi(a) = b$*.*

Proof. It is not difficult to explicitly construct such a homeomorphism when $\mathbb{G}$ is a ball in $\mathbb{R}^n$. We now connect a and b via a sequence of points $a = a_1, a_2, ..., a_{N+1} = b$ and balls $\mathbb{B}_1, \mathbb{B}_2, ..., \mathbb{B}_N$ such that

$$a_i, a_{i+1} \in \mathbb{B}_i \subset \mathbb{G} \qquad i = 1, 2, ..., N$$

Let $\Psi_i : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism which is the identity on $\mathbb{R}^n \setminus \mathbb{B}_i$ and takes a_i into a_{i+1}. Then $\Psi = \Psi_1 \circ \Psi_2 \circ ... \circ \Psi_N$ is the desired map.

In the sequel $\mathbb{G}$ will be the unbounded component of $\mathbb{R}^n \setminus \Gamma_\lambda$. Choose and fix $\delta \in (0, \lambda)$. In view of the Supposition we find a point $y_\circ \in \Delta^\lambda_\delta = h(\mathbf{B}_\lambda \setminus \overline{\mathbf{B}_\delta}) \subset h(\tilde{\mathbf{B}}_\lambda) \subset \mathbb{G}$. Then, with the aid of Lemma 6.4, we move $y_\circ$ far away from the boundary of $\mathbb{G}$. Precisely, we construct a homeomorphism $\Psi : \mathbb{R}^n \xrightarrow{\text{onto}} \mathbb{R}^n$ which equals the identity in $\mathbb{R}^n \setminus \mathbb{G}$ and satisfies

$$\operatorname{dist}(\Psi(y_\circ), \Gamma_\lambda) \geqslant \operatorname{diam} \Gamma_\lambda \tag{48}$$

This inequality also holds for the subset $\mathbf{S} = \Gamma_\lambda \setminus \tilde{\Gamma}_\lambda$ in place of Γ_λ,

$$\operatorname{dist}(\Psi(y_\circ), \mathbf{S}) \geqslant \operatorname{diam} \mathbf{S} \tag{49}$$

Let us conveniently choose a coordinate system in the target space so that $\Psi(y_\circ)$ will be the origin. This realization is particularly convenient. For that, (49) tells us that no straight line through the origin can intersect $\mathbf{S}$ on both sides. Precisely, we have

$$\frac{\xi_1}{|\xi_1|} + \frac{\xi_2}{|\xi_2|} \neq 0 \qquad \text{whenever } \xi_1, \xi_2 \in \mathbf{S} \tag{50}$$

Note that

$$0 = \Psi(y_\circ) \in \Psi(\Delta^\lambda_\delta) \subset \mathbb{G} \tag{51}$$

We fix a ball $\mathbb{B} = \{z \in \mathbb{R}^n ; \;\; |z| < r\} \subset \Psi(\Delta^\lambda_\delta)$. It follows from Lemma 6.3 that the set Δ^λ_ϵ lies in the bounded component of $\mathbb{R}^n \setminus \Gamma_\epsilon$, so does the subset $\Delta^\lambda_\delta \subset \Delta^\lambda_\epsilon$. In particular, $\mathbb{B}$ is contained in the bounded component of $\mathbb{R}^n \setminus \Psi(\Gamma_\epsilon)$, for every $0 < \epsilon < \delta$. Let Ω_ϵ denote this bounded component,

$$\mathbb{B} \subset \Omega_\epsilon \subset \mathbb{R}^n \setminus \Psi(\Gamma_\epsilon)\,, \qquad \text{for every } 0 < \epsilon < \delta \tag{52}$$

Concerning the boundary of Ω_ϵ, we observe that

$$\partial\Omega_\epsilon \subset \Psi(\Gamma_\epsilon) = \Psi(\tilde{\Gamma}_\epsilon) \cup \Psi(\mathbf{S}) = \Psi(\tilde{\Gamma}_\epsilon) \cup \mathbf{S} \tag{53}$$

Next we define a map $\Phi : \partial\Omega_\epsilon \to \mathbb{R}^{n-1}$. Let $\pi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ denote the projection; $h(x) = \omega$ for $x = (\omega, z)$. Then for $\xi \in \Psi(\tilde{\Gamma}_\epsilon)$ we set $\Phi(\xi) =$

$\left(1-|\omega|^2\right)(\omega,1) \in \mathbb{R}^{k+1} \subset \mathbb{R}^{n-1}$, where $\omega = \pi \circ h^{-1} \circ \Psi^{-1}(\xi)$. For $\xi \in \mathbf{S}$, we set $\Phi(\xi) = 0$. In the first case $\Phi : \Psi(\tilde{\Gamma_\epsilon}) \to \mathbb{R}^{k+1}$ is continuous. Moreover, since $x = h^{-1} \circ \Psi^{-1}(\xi) = (\omega, z) \in \tilde{\mathbb{S}_\epsilon}$, we have the relation $|z|^2 = \epsilon^2\left(1-|\omega|^2\right) > 0$. Hence $\Phi(\xi) \neq 0$. Actually the map $\Phi : \partial\Omega_\epsilon \to \mathbb{R}^{n-1}$ remains continuous on the entire boundary of Ω_ϵ. To see this we consider points $\xi_i \in \Psi(\tilde{\Gamma_\epsilon})$ approaching $\mathbf{S}$. We need to show that $\lim\limits_{i\to\infty} \Phi(\xi_i) = 0$. Following the definition of $\Phi(\xi_i)$, we obtain the points $x_i = h^{-1} \circ \Psi(\xi_i) \in \tilde{\mathcal{S}_\epsilon}$ which approach the crack $\mathfrak{X}$. We write those points as $x_i = (\omega_i, z_i)$, then $z_i \to 0$. Concerning the first coordinate, since $x_i \in \mathcal{S}_\epsilon$ we see that $|\omega_i| \to 1$. Then we conclude that $\Phi(\xi_i) = \left(1-|\omega_i|^2\right)(\omega_i, 1) \to 0$ as desired.

In our final step in the proof of Proposition 6.1 we shall appeal to Joshi's Antipodal Theorem in the domain $\Omega_\epsilon \subset \mathbb{R}^n$ and $\Phi : \partial\Omega_\epsilon \to \mathbb{R}^{n-1}$. Accordingly, there exist two points $\xi_1, \xi_2 \in \partial\Omega_\epsilon$ such that

$$\Phi(\xi_1) = \Phi(\xi_2) \qquad \text{and} \qquad \frac{\xi_1}{|\xi_1|} + \frac{\xi_2}{|\xi_2|} = 0 \tag{54}$$

Since ξ_1 and ξ_2 lie on the opposite sides of a straight line passing through the origin of the ball $\mathbb{B}$, we have

$$|\xi_1 - \xi_2| \geqslant \operatorname{diam} \mathbb{B} = 2r\,, \qquad \text{independently of } \epsilon \to 0 \tag{55}$$

Observe that the common value $\Phi(\xi_1) = \Phi(\xi_2)$ cannot be equal to zero, since otherwise we would have $\xi_1, \xi_2 \in \mathbf{S}$, contradicting (50). This leaves us with the only possibility that $\xi_i \in \Psi(\tilde{\Gamma_\epsilon})$ for $i = 1, 2$. Following the definition of $\Phi(\xi_i)$ we obtain the points $x_i = h^{-1} \circ \Psi^{-1}(\xi_i) \in \tilde{\mathcal{S}_\epsilon}$ and $\Phi(\xi_i) = \left(1-|\omega_i|^2\right)(\omega_i, 1) \in \mathbb{R}^{k+1}$. Here ω_i stands for the component of $x_i = (\omega_i, z_i)$. The equation $\Phi(\xi_1) = \Phi(\xi_2)$ simply means that

$$\left(1-|\omega_1|^2\right)(\omega_1, 1) = \left(1-|\omega_2|^2\right)(\omega_2, 1) \neq 0 \tag{56}$$

Hence $\omega_1 = \omega_2$. In other words, the points x_1, x_2 not only belong to $\mathcal{S}_\epsilon$ but also have the same projection into $\mathfrak{X}$. Thus, they belong to the same sphere $\mathbb{S}^{\ell-1}(a, \rho)$, where $a = (\omega, 0) = (\omega_1, 0) = (\omega_2, 0)$ and $\rho = \epsilon\sqrt{1-|\omega|^2}$. The torus $\mathbb{T}_\rho$, see (34), lies in $\mathbb{X} \setminus \mathfrak{X}$, at least when ϵ is sufficiently small. Now the oscillation inequality (35) tells us that $|h(x_1) - h(x_2)|$ is arbitrarily small when ϵ approaches zero. On the other hand

$$2r \leqslant |\xi_1 - \xi_2| = |\Psi\big(h(x_1)\big) - \Psi\big(h(x_2)\big)| \tag{57}$$

which contradicts continuity of Ψ. This contradiction completes the proof of Proposition 6.1.

6.4 The final step of the proof of Theorem 1.1

In view of Lemma 2.3 we assume that $\mathfrak{X} = \mathbb{X} \cap \mathbb{R}^k$. It suffices to show that $h : \mathbb{X} \setminus \mathfrak{X} \to \mathbb{R}^n$ is locally uniformly continuous. We shall actually show that for every pair $a, b \in \mathbb{X} \setminus \mathfrak{X}$ it holds

$$|h(a) - h(b)|^n \preccurlyeq \int_{B(x_\circ, 2\epsilon)} \| Dh \|^n , \quad x_\circ = \frac{a+b}{2} \text{ and } \epsilon = \frac{|a-b|}{2} \tag{58}$$

provided $B(x_\circ, 2\epsilon) \subset \mathbb{X}$. This, in view of equiintegrability of $\| Dh \|^n$, yields uniform continuity of h on $\mathbb{F} \setminus \mathfrak{X}$, for every compact $\mathbb{F} \subset \mathbb{X}$. Consequently, it yields unique continuous extension of f to $\mathbb{F}$.

We shall consider h on $\mathbb{X}$ as an element of Sobolev class $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$ defined everywhere. In fact, any extension of $h : \mathbb{X} \setminus \mathfrak{X} \to \mathbb{R}^n$ lies in $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$. This is easy to see, for instance, by applying the ACL characterization of Sobolev spaces. The k-flatness of $\mathfrak{X}$, $1 \leqslant k \leqslant n-2$, is critical for this observation.

We consider the concentric spheres $\mathbb{S}_t = \{x \,;\ \ |x - x_\circ| = t\}$, $\epsilon < t < 2\epsilon$, to show that

$$|h(a) - h(b)|^n \preccurlyeq t \int_{\mathbb{S}_t} \| Dh \|^n \tag{59}$$

for almost every t in the above interval. We choose and fix those parameters t for which h restricted to $\mathbb{S}_t$ belongs to the Sobolev class $\mathscr{W}^{1,n}(\mathbb{S}_t, \mathbb{R}^n)$. This is immediate from the Sobolev imbedding theorem on spheres that h, being continuous on $\mathbb{S}_t \setminus \mathfrak{X}$, admits unique continuous extension to $\mathbb{S}_t$. Moreover, upon such extension, we have

$$\left[\operatorname*{osc}_{\mathbb{S}_t} h \right]^n \preccurlyeq t \int_{\mathbb{S}_t} \| Dh \|^n \tag{60}$$

We are now in a position to apply Proposition 6.1. Of course the unit sphere $\mathcal{S}_1$ in this proposition can be replaced by any sphere. Accordingly, the points $h(a)$ and $h(b)$ lie in the bounded component of $\mathbb{R}^n \setminus h(\mathbb{S}_t)$. The geometric arguments in (30) yield,

$$|h(a) - h(b)|^n \leqslant [\operatorname{diam} h(\mathbb{S}_t)]^n \preccurlyeq t \int_{\mathbb{S}_t} \| Dh \|^n \tag{61}$$

Upon integration with respect to t we obtain

$$|h(a) - h(b)|^n = \frac{1}{\log 2} \int_\epsilon^{2\epsilon} \frac{|h(a) - h(b)|^n \, \mathrm{d}t}{t} \preccurlyeq \int_{B(x_\circ, 2\epsilon)} \| Dh \|^n$$

as claimed.

As for the Lusin's condition $\mathscr{N}$, we argue as follows. Given concentric balls $B \subset 2B \subset \mathbb{X}$, we have

$$|f(B)| \preccurlyeq \int_{2B} \| Dh \|^n \tag{62}$$

This is immediate from (58). Let $\mathbb{E} \subset \mathbb{X}$ be a set of zero measure, we imbed $\mathbb{E}$ in an open set $\mathbb{U} \subset \mathbb{X}$ whose measure can be arbitrarily small. Next, with the aid of Whitney's decomposition of $\mathbb{U}$ we find a cover of $\mathbb{U}$, $\mathbb{U} = \cup B_i$, in such a way that the overlapping number depends only on the dimension n,

$$\sum_{i=1}^{\infty} \chi_{2B_i}(x) \leqslant N = N(n)$$

Then we conclude that

$$|h(\mathbb{E})| \leqslant |h(\mathbb{U})| \leqslant \sum |h(B_i)| \preccurlyeq \sum \int_{2B_i} \| Dh \|^n \preccurlyeq \int_{\mathbb{U}} \| Dh \|^n$$

where the last integral can be made arbitrarily small. The proof of Theorem 1.1 is complete.

7 Proof of Theorem 1.2

As in the proof of Theorem 1.1 it suffices to establish local uniform continuity of $h : \mathbb{X} \setminus \mathfrak{X} \to \mathbb{R}^n$. In this case we have the Sobolev regularity $h \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$ for granted. We choose and fix points

$$a, b \in B(x_\circ, \epsilon) \subset B(x_\circ, 2\epsilon) \subset \mathbb{X}$$

where $2x_\circ = a + b$ and $2\epsilon = |a - b|$. As before, for almost every $\epsilon < t < 2\epsilon$ we have the trace of h on the sphere $\mathbb{S}_t = \{x\,;\ \ |x - x_\circ| = t\}$ in the Sobolev space $\mathscr{W}^{1,n}(\mathbb{S}_t, \mathbb{R}^n)$. In addition to this, we will require that $\mathfrak{X} \cap \mathbb{S}_t$ has zero $(n-1)$-measure. This is possible for almost every t, by Fubini's theorem. Moreover, $h : \mathbb{S}_t \to \mathbb{R}^n$ is continuous and satisfies

$$\left[\operatorname*{osc}_{\mathbb{S}_t} h\right]^n \preccurlyeq t \int_{\mathbb{S}_t} \| Dh \|^n \tag{63}$$

Theorem 3.6 tells us that the set $h(\mathbb{S}_t)$ separates $\mathbb{R}^n$. Now it remains to show that $h(a)$ and $h(b)$ do not belong to the unbounded component of $h(\mathbb{S}_t)$. It is at this point that our argument differs from the proof of Theorem 1.1; we cannot apply Proposition 6.1. However, the assumption that the set $h\{\mathfrak{X}\}$ does not separate domains comes to the rescue.

Assume to the contrary, that $h(a)$ lies in the unbounded component of $\mathbb{R}^n \setminus h(\mathbb{S}_t)$, which we denote by $\mathbb{V}$. The set $\mathbb{V} \setminus h\{\mathfrak{X}\}$ is also connected. Let $\gamma \subset \mathbb{V} \setminus h\{\mathfrak{X}\}$ denote the infinite path connecting $h(a)$ with ∞. The open set $h(\mathbb{X} \setminus \mathfrak{X})$ (not necessarily connected) is bounded, so γ intersects its boundary. We denote by γ' the component of $\gamma \cap h(\mathbb{X} \setminus \mathfrak{X})$ which contains $h(a)$. Its closure intersects the boundary of $h(\mathbb{X} \setminus \mathfrak{X})$. Precisely, we have a sequence of points $y_i \in \gamma'$ such that $\operatorname{dist}\big(y_i, \partial h(\mathbb{X} \setminus \mathfrak{X})\big) \to 0$. Consider the preimages $x_i = h^{-1}(y_i) \in \mathbb{X} \setminus \mathfrak{X}$. Note that $h^{-1}(\gamma')$ is connected and disjoint with $\mathbb{S}_t$. Since it contains a the entire set $h^{-1}(\gamma')$ must lie inside the sphere $\mathbb{S}_t$ and, consequently, stays away from the boundary of $\mathbb{X}$. On the other hand x_i approach the boundary of $\mathbb{X} \setminus \mathfrak{X}$ but not $\mathfrak{X}$. This is a contradiction. For Lusin's condition $\mathscr{N}$ we argue as in Theorem 1.1; that is, Inequality (63) implies (58). The rest of the arguments is the same, completing the proof of Theorem 1.2.

8 Proof of Theorem 1.3

We may assume that $\mathfrak{X}$ is a flat k-dimensional disk,

$$\mathfrak{X} = \left\{ x = (x_1, ..., x_k, 0, ..., 0)\,; \quad x_1^2 + ... + x_k^2 \leqslant \frac{1}{2} \right\}$$

and that the unit ball of $\mathbb{R}^n$ lies in $\mathbb{X}$. This can be achieved by a suitable topological change of variables in $\mathbb{X}$. In particular, all the ellipsoids $\mathcal{S}_\lambda$, $0 < \lambda \leqslant 1$ together with their interiors $\mathbf{B}_\lambda$ lie in $\mathbb{X}$. In what follows we shall let λ go to zero so that the ellipsoids approach the crack $\mathfrak{X}$. As for the configuration $h(\mathfrak{X})$ let us assume, to the contrary of the statement in Theorem 1.3, that $\mathbb{R}^n \setminus \mathbb{V}$ contains a k-separator, where $\mathbb{V}$ is the unbounded component of $\mathbb{R}^n \setminus h(\mathfrak{X})$. It involves no loss of generality in assuming that $\mathbb{R}^n \setminus \mathbb{V}$ actually contains a continuum $\mathbb{K} \subset \mathbb{R}^{k+1}$, separating $\mathbb{R}^{k+1}$ and that the point $0 \in \mathbb{R}^{k+1}$ lies in a bounded component of $\mathbb{R}^{k+1} \setminus \mathbb{K}$. The sets $\Gamma_\lambda = h(\mathcal{S}_\lambda)$ are topological $(n-1)$-spheres separating $\mathbb{R}^n$. By Proposition 6.1 the set $h(\mathfrak{X})$ lies inside Γ_λ for every $0 < \lambda \leqslant 1$. In particular $\mathbb{K}$ lies inside Γ_λ. Denote by $\Omega_\lambda = \mathbb{R}^{k+1} \cap \mathbb{U}_\lambda$, where $\mathbb{U}_\lambda$ stands for the bounded component of $\mathbb{R}^n \setminus \Gamma_\lambda$. Thus $\partial\Omega_\lambda = \mathbb{R}^{k+1} \cap \Gamma_\lambda \subset \mathbb{R}^{k+1}$. Next, we define a continuous map

$$\Phi = \pi \circ h^{-1} \,:\, \partial\Omega_\lambda \to \mathbb{R}^k \tag{64}$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^k$ denotes the projection of a point $x = (\omega, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, $\pi(x) = \omega$, see Section 6.2. At this stage we appeal to Joshi's Antipodal Theorem

3.8. Accordingly, there exist antipodal points $\xi_1, \xi_2 \in \partial\Omega_\lambda$ such that

$$\frac{\xi_1}{|\xi_1|} + \frac{\xi_2}{|\xi_2|} = 0 \qquad \text{and} \qquad \Phi(\xi_1) = \Phi(\xi_2)$$

The first equation yields

$$2 \operatorname{dist}(0, \mathbb{K}) \leqslant |\xi_1 - \xi_2|$$

by an elementary geometric consideration. The second equation tells us that $\pi(x_1) = \pi(x_2)$, where $x_1 = h^{-1}(\xi_1)$ and $x_2 = h^{-1}(\xi_2)$. Writing these points as $x_1 = (\omega, z_1)$ and $x_2 = (\omega, z_2)$, with $\omega = \pi(x_1) = \pi(x_2)$, we observe that

$$|x_1 - x_2| = |z_1 - z_2| \leqslant |z_1| + |z_2|$$

On the other hand, since x_1 and x_2 lie in the same ellipsoid $\mathcal{S}_\lambda$, we see that $|z_1| = |z_2| = \lambda\sqrt{1 - |\omega|^2} \leqslant \lambda$. This means that $|x_1 - x_2| \leqslant 2\lambda \to 0$, because we let λ go to zero. In conclusion,

$$0 < 2 \operatorname{dist}(0, \mathbb{K}) \leqslant |\xi_1 - \xi_2| = |h(x_1) - h(x_2)| \to 0$$

by uniform continuity of $h : \overline{\mathbb{B}} \to \mathbb{R}^n$. This contradiction proves that $\mathbb{R}^n \setminus \mathbb{V}$ does not contain any k-separator.

It remains to show that $h(\mathbb{X})$ is an open set, as it is obviously connected. We note that Υ has no interior and does not separate $\mathbb{R}^n$, otherwise $\mathbb{R}^n \setminus \mathbb{V}$ would contain a k-separator. We now recall the sets $\tilde{\mathbf{B}}_\lambda = \mathbf{B}_\lambda \setminus \mathfrak{X}$. According to Proposition 6.1, we see that $h(\tilde{\mathbf{B}}_\lambda) \subset \mathbb{U}_\lambda \setminus \Upsilon$. We aim to show that $h(\tilde{\mathbf{B}}_\lambda) = \mathbb{U}_\lambda \setminus \Upsilon$. For this let us note that both sets have the same boundary

$$\partial h(\tilde{\mathbf{B}}_\lambda) = h\{\partial \tilde{\mathbf{B}}_\lambda\} = \Gamma_\lambda \cup \Upsilon = \partial(\mathbb{U}_\lambda \setminus \Upsilon) \tag{65}$$

because Υ is a compact subset of $\mathbb{U}_\lambda$ without interior. Also notice that $\mathbb{U}_\lambda \setminus \Upsilon = \mathbb{U}_\lambda \cap (\mathbb{R}^n \setminus \Upsilon)$, where both $\mathbb{U}_\lambda$ and $\mathbb{R}^n \setminus \Upsilon$ are domains covering $\mathbb{R}^n$. By virtue of unicoherence of $\mathbb{R}^n$, Theorem 3.1, $\mathbb{U}_\lambda \setminus \Upsilon$ is a domain. Recall a very general fact that if a nonempty subset A of a domain B has boundary in ∂B, then $A = B$. Applying it to $A = h(\tilde{\mathbf{B}}_\lambda)$ and $B = \mathbb{U}_\lambda \setminus \Upsilon$, (65) yields $h(\tilde{\mathbf{B}}_\lambda) = \mathbb{U}_\lambda \setminus \Upsilon$. We conclude that

$$h(\mathbb{X}) = h(\mathbb{X} \setminus \mathfrak{X}) \cup h(\tilde{\mathbf{B}}_\lambda) \cup h(\mathfrak{X}) = h(\mathbb{X} \setminus \mathfrak{X}) \cup \mathbb{U}_\lambda \,, \qquad \text{for every } 0 < \lambda < 1$$

Thus $h(\mathbb{X})$ is an open set. Finally, we modify $h : \mathbb{X} \to \mathbb{R}^n$ inside the ellipsoid $\mathcal{S}_\lambda$, $0 < \lambda < 1$. The Schönflies Theorem 3.7, tells us that $h_\lambda : \mathcal{S}_\lambda \to \mathbb{R}^n$ can be homeomorphically extended as $h_\lambda : \tilde{\mathbf{B}}_\lambda \xrightarrow{\text{onto}} \tilde{\mathbb{U}}_\lambda$. Setting

$$H(x) = \begin{cases} h(x) & \text{for } x \in \mathbb{X} \setminus \mathbf{B}_\lambda \\ h_\lambda & \text{for } x \in \mathbf{B}_\lambda \end{cases} \tag{66}$$

we obtain a homeomorphism $H : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$. Thus $\mathbb{X}$ and $\mathbb{Y}$ are the same topological type, completing the proof of Theorem 1.3.

9 An Example

We shall illustrate that a flat crack can be squeezed into a point by a deformation of finite conformal energy. However, Proposition 1.2 tells us that it would be impossible to deform it back with finite total energy. The example we are recalling, due to J.M. Ball [6], also shows that the continuous extension to a map $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ neither be discrete nor open. It remains unclear, however, as to whether h is monotone in the sense of C.B. Morrey [60]. For an excellent old survey about monotone mappings we refer to L. F. McAuley [59].

Consider the cylindrical domain as a reference configuration,

$$\mathbb{X} = \mathbb{B}^{n-1} \times (-2,2) = \big\{ x = (\omega, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \, ; \ \ |\omega| < 1 \, , \ |t| < 2 \big\}$$

and the crack in $\mathbb{X}$,

$$\mathfrak{X} = \big\{ \, (0,t) \in \mathbb{R}^{n-1} \times \mathbb{R} \, ; \ \ -1 \leqslant t \leqslant 1 \big\}$$

As for the deformed configuration we consider the same cylinder $\mathbb{Y} = \mathbb{X}$, but with a puncture at the origin; that is, $\Upsilon = \{(0,0)\}$. Then the mapping $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ we are referring to takes the form

$$h(x) = \begin{cases} (\, \omega, \, t \, |\omega| \,) & |t| \leqslant 1 \\ \Big(\omega, \, \frac{t}{|t|} \left[\, 2\,|t| - 2 + 2\,|\omega| - |t|\,|\omega| \, \right] \Big) & 1 \leqslant |t| < 2 \end{cases} \tag{67}$$

It is easy to see that $h \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$. It is a homeomorphism of $\mathbb{X} \setminus \mathfrak{X}$ onto $\mathbb{Y} \setminus \Upsilon$ and $h(\mathfrak{X}) = \Upsilon$. However, it takes the open cylinder $\mathbb{X}' = \mathbb{B}^{n-1} \times (-1,1)$ onto a clepsydra

$$\mathbb{Y}' = \{ \, (\omega, t) \, ; \ \ |t| < |\omega| < 1 \, , \quad \text{or} \quad t = \omega = 0 \}$$

which is obviously connected but not open, see Figure 1. The preimage of the origin is not discrete, nevertheless, connected. In fact h is monotone; preimage of every continuum is continuum.

Remark 9.1. The boundary of $\mathbb{X}'$ is a topological sphere, while $h : \partial \mathbb{X}' \to \mathbb{R}^n$ is continuous and injective upon removal of two points $(0, \pm 1)$. The set $h(\partial \mathbb{X}') = \partial \mathbb{Y}'$ separates $\mathbb{R}^n$, in agreement with Theorem 3.6.

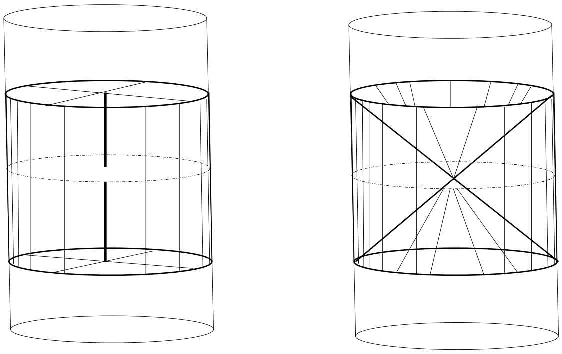

Figure 1: Squeezing a line segment into a point.

10 Zero Infimum Energy

When $p \geqslant n$ the infimum of the p-harmonic energy is always positive

$$\begin{aligned} \mathcal{E}_p[h] &= \int_{\mathbb{X}} \| Dh(x) \|^p \, \mathrm{d}x \geqslant |\mathbb{X}|^{1-\frac{p}{n}} \left(\int_{\mathbb{X}} \| Dh(x) \|^n \, \mathrm{d}x \right)^{\frac{p}{n}} \\ &\geqslant n^{\frac{p}{2}} \, |\mathbb{X}|^{1-\frac{p}{n}} \left(\int_{\mathbb{X}} J(x,h) \, \mathrm{d}x \right)^{\frac{p}{n}} = n^{\frac{p}{2}} \, |\mathbb{X}|^{1-\frac{p}{n}} \, |\mathbb{Y}|^{\frac{p}{n}} \end{aligned} \tag{68}$$

Before passing to the case $p < n$ let us look more closely at the minimizers of the n-harmonic energy.

Example 10.1. (Conformal automorphisms of the unit ball) To every point $a \in \mathbb{B} \subset \mathbb{R}^n$, $a \neq 0$, there corresponds a conformal deformation $h_a : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}$, defined by

$$h_a(x) = \tilde{a} + \frac{\left(|\tilde{a}|^2 - 1\right)(x - \tilde{a})}{|x - \tilde{a}|^2}\,, \quad \tilde{a} = \frac{a}{|a|^2}\,, \quad h_a(0) = a\,,\ h_a(a) = 0 \tag{69}$$

All these mappings share the same conformal energy. Indeed, because of conformality, we have

$$\int_{\mathbb{B}} \| Dh_a(x) \|^n \, \mathrm{d}x \ = \ n^{\frac{n}{2}} \int_{\mathbb{B}} J(x, h_a) \, \mathrm{d}x = n^{\frac{n}{2}} \, |\mathbb{B}| \tag{70}$$

Whenever a goes to a boundary point $a_\circ \in \partial\mathbb{B}$, we find that $\{h_a\}$ converges c-uniformly (uniformly on compact subsets) to the constant mapping $f(x) \equiv a_\circ$.

More refined computation shows that

$$\inf \int_{\mathbb{B}} \| Dh(x) \|^p \, \mathrm{d}x = \lim_{a \to a_\circ} \int_{\mathbb{B}} \| Dh_a(x) \|^p \, \mathrm{d}x = 0 \qquad 1 \leqslant p < n$$

10.1 Mappings that keep one point fixed

The situation is quite different if one requires that all mappings $h : \mathbb{X} \to \mathbb{Y}$ leave a given point $a \in \mathbb{X}$ fixed, or take it into another given point. Let us first examine two special subclasses of such mappings.

10.1.1 Radial deformations

The simplest example is that of radial mappings $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}$ in $\mathscr{W}^{1,1}(\mathbb{B}, \mathbb{B})$ which keep the origin fixed.

THEOREM 10.1. (Radial stretching) *Let* $h(x) = R(|x|)\,\frac{x}{|x|}$ *with* $|x| < 1$ *, where* $R : [0,1] \to [0,1]$ *is continuously increasing from* $R(0) = 0$ *to* $R(1) = 1$. *Then we have the following sharp lower bound of the p-harmonic energy.*

$$\left(\frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} \|\, Dh(x) \,\|^{\,p}\, \mathrm{d}x\right)^{\frac{1}{p}} \geqslant \frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} \|\, Dh(x) \,\|\, \mathrm{d}x \geqslant \sqrt{n}\,, \qquad p \geqslant 1 \tag{71}$$

Equality occurs if and only if $h(x) \equiv x$.

Proof. Denote $r = |x|$. We begin with an elementary point-wise estimate,

$$\frac{1}{\sqrt{n}}\,\|\, Dh(x) \,\| = \sqrt{\frac{1}{n}\dot{R}^2 + \left(1 - \frac{1}{n}\right)\frac{R^2}{r^2}} \geqslant \frac{1}{n}\dot{R} + \left(1 - \frac{1}{n}\right)\frac{R}{r}$$

which is immediate by convexity of the squaring function. Hence

$$\begin{aligned} \frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} \frac{\|\, Dh(x) \,\|}{\sqrt{n}}\, \mathrm{d}x \;&\geqslant\; \int_0^1 \dot{R}\, r^{n-1}\, \mathrm{d}r + (n-1)\int_0^1 R\, r^{n-2}\, \mathrm{d}r \\ &=\; \int_0^1 \left(r^{n-1}R\right)'\, \mathrm{d}r = 1 \end{aligned}$$

Clearly, equality occurs if and only if $R(r) = r$.

Remark 10.1. It follows that the lower bound (71) remains true with $\sqrt{n}$ replaced by 1 if the Hilbert-Schmidt norm is replaced by the operator norm.

Radial deformations in nonlinear elasticity are discussed in [72] and [40].

10.1.2 Spherical deformations

The result is different if we minimize the energy in a slightly larger class of the so-called spherical mappings. A homeomorphism $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}$ is called spherical map if it takes spheres (centered at the origin) into spheres. Analytically it means that $|h|$ depends only on $|x|$.

THEOREM 10.2. (Energy of spherical mappings) *Let $\mathscr{S}$ denote the class of spherical mappings $h : \mathbb{B} \to \mathbb{B}$ of Sobolev class $\mathscr{W}^{1,p}(\mathbb{B}, \mathbb{B})$, $h(0) = 0$ and $h(1) = 1$. Then*

$$\inf_{h \in \mathscr{S}} \int_{\mathbb{B}} \| Dh(x) \|^p \, \mathrm{d}x = \begin{cases} 0\,, & \text{if } 1 \leqslant p < n-1 \\ \textit{positive}\,, & \text{if } p \geqslant n-1 \end{cases} \tag{72}$$

Proof. First we demonstrate that the infimum equals zero for $1 \leqslant p < n-1$. Preceding the proof we discuss mappings of the unit sphere. The general problem to be addressed here is the following. Among all homeomorphisms $\Phi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ of Sobolev class $\mathscr{W}^{1,p}(\mathbb{S}^{n-1}, \mathbb{S}^{n-1})$ find the one that minimizes the spherical average integral

$$\mathcal{E}_p[\Phi] = \fint_{\mathbb{S}^{n-1}} \left[\alpha^2 + \| D\Phi \|^2\right]^{\frac{p}{2}} = \fint_{\mathbb{S}^{n-1}} \left[\alpha^2 + (n-1) | D\Phi |^2\right]^{\frac{p}{2}} \tag{73}$$

where α is any given number. Here we introduced the mean Hilbert-Schmidt norm of the linear tangent map

$$D\Phi : \mathbf{T}_x \, \mathbb{S}^{n-1} \to \mathbf{T}_y \, \mathbb{S}^{n-1}\,, \qquad y = \Phi(x) \tag{74}$$

by the rule

$$(n-1)|D\Phi|^2 = \mathrm{Tr}\left[\, D^*\Phi \;\; D\Phi \,\right]\,, \tag{75}$$

so that $|D\Phi| = 1$ for the identity map. One particular question to ask is whether the identity map $id : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is a minimizer. It is tempting to apply Jensen's inequality.

$$\mathcal{E}_p[\Phi] \geqslant \left[\alpha^2 + (n-1) \fint_{\mathbb{S}^{n-1}} |D\Phi|^2\right]^{\frac{p}{2}}\,, \qquad p \geqslant 2 \tag{76}$$

Then in dimensions $n = 2$ and $n = 3$, one can invoke Hölder's and Hadamard's inequality to obtain

$$\fint_{\mathbb{S}^{n-1}} |D\Phi|^2 \geqslant \left(\fint_{\mathbb{S}^{n-1}} |D\Phi|^{n-1}\right)^{\frac{2}{n-1}} \geqslant \left(\fint_{\mathbb{S}^{n-1}} J(x, \Phi)\, dx\right)^{\frac{2}{n-1}} = 1 \tag{77}$$

The Jacobian determinant $J(x, \Phi)$ is point-wise controlled by $|D\Phi(x)|^{n-1}$. Then $J(x, \Phi)\, \mathrm{d}x$ is the pullback via Φ of the standard $(n-1)$-form Ξ on $\mathbb{S}^{n-1}$, see (20). The last step is justified by the fact that

$$\fint_{\mathbb{S}^{n-1}} J(x, \Phi)\, dx = \fint_{\mathbb{S}^{n-1}} \Phi^\sharp(\Xi) = \deg \Phi = 1 \tag{78}$$

In dimensions greater than 3, however, the infimum of the Dirichlet integrals

$$\inf \int_{\mathbb{S}^{n-1}} |D\Phi|^2 \;; \quad \Phi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1} \tag{79}$$

equals zero, so is not attained. For computational convenience we shall enlarge the class of contenders. A Lipschitz map $\Phi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ will be called *admissible* if it can be approximated by Lipschitz homeomorphism $\Phi_i : \mathbb{S}^{n-1} \xrightarrow{\text{onto}} \mathbb{S}^{n-1}$, uniformly together with the derivatives. That is, $\Phi_i \rightrightarrows \Phi$ and $D\Phi_i \rightrightarrows D\Phi$. The infimum in (79) remains unchanged if we replace homeomorphisms $\Phi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ by admissible mappings.

Consider a spherical cap $\mathbb{S}^\epsilon \subset \mathbb{S}^{n-1}$ of small radius ϵ and with center at the north pole, $0 < \epsilon < \pi$. A mapping $\Phi^\epsilon : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ which stretches $\mathbb{S}^\epsilon$ (uniformly along the longitude lines) up to the south pole and takes the rest of the sphere into the south pole, is certainly admissible. Elementary geometric considerations show that

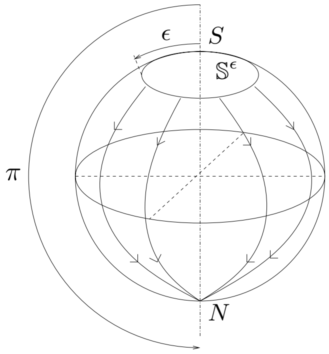

Figure 2: Small spherical cap is stretched uniformly onto the entire sphere.

$$|D\Phi^\epsilon| \leqslant \begin{cases} \frac{\pi}{\epsilon} & \text{in } \mathbb{S}^\epsilon \quad \text{and} \quad |\mathbb{S}^\epsilon| \approx \epsilon^{n-1} \\ 0 & \text{otherwise} \end{cases} \tag{80}$$

Hence, for every $1 \leqslant p < n-1$ we find that

$$\fint_{\mathbb{S}^{n-1}} |D\Phi^\epsilon|^p = O(\epsilon^{n-1-p}) \to 0 \quad \text{as } \epsilon \to 0 \tag{81}$$

With these spherical maps at hand, we construct deformations $h : \mathbb{B} \to \mathbb{B}$, $h(0) = 0$, having arbitrarily small p-harmonic energy, $1 \leqslant p < n-1$,

$$h(x) = R_\epsilon\big(|x|\big)\Phi^\epsilon\left(\frac{x}{|x|}\right), \quad \text{where } R_\epsilon(t) = \begin{cases} \frac{1}{\epsilon}\,t\,, & \text{for } 0 \leqslant t \leqslant \epsilon \\ 1\,, & \text{for } \epsilon \leqslant t \leqslant 1 \end{cases} \tag{82}$$

Obviously, h is an admissible contender for the infimum at (72). The Hilbert-

Schmidt norm of its differential takes the form

$$\| Dh(x) \|^2 = \left| \dot{R}_\epsilon(r) \right|^2 + (n-1) \left(\frac{R_\epsilon(r)}{r} \right)^2 \left| D\Phi^\epsilon(\omega) \right|^2 , \qquad \text{for } x = r\omega$$

Hence, for $1 \leqslant p < n-1$, we obtain the desired estimate

$$\begin{aligned} \int_{\mathbb{B}} \| Dh(x) \|^p \, \mathrm{d}x &\preccurlyeq \int_0^1 \left| \dot{R}_\epsilon(r) \right|^p r^{n-1} \, \mathrm{d}r + \left(\int_0^1 r^{n-1-p} \, \mathrm{d}r \right) \int_{\mathbb{S}^{n-1}} |D\Phi^\epsilon|^p \\ &= \frac{1}{n} \epsilon^{n-p} + \frac{1}{n-p} O(\epsilon^{n-1-p}) \to 0 \qquad \text{as } \epsilon \to 0 \end{aligned}$$

because the implied constant is independent of ϵ.

We now proceed to the case $p \geqslant n-1$ in (72). By virtue of Hölder's inequality it suffices to consider the borderline case $p = n-1$. Be aware that it is not enough to test the infimum at (72) with mappings of the form $R\big(|x|\big)\Phi\left(\frac{x}{|x|}\right)$, $\Phi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. These are only special cases of spherical mappings. For full generality Φ must depend on $r = |x|$ as well. We begin with the most general expression for the Hilbert-Schmidt norm of Dh,

$$\| Dh \|^2 = |h_r|^2 + (n-1) |h_\omega|^2 \tag{83}$$

where h_r and h_ω stand for the radial and spherical derivatives of h, see (24) and (25). The following elementary inequality, simple though it is, will lead us to sharp results,

$$\left[x^2 + (n-1) y^2 \right]^{\frac{n-1}{2}} \geqslant \alpha_n \left[x^{n-1} + (n-1) y^{n-1} \right] \tag{84}$$

for $x, y \geqslant 0$, where

$$\alpha_n = \begin{cases} \frac{\sqrt{2}}{2} , & \text{if } n = 2 \\ 1 , & \text{if } n \geqslant 3 \end{cases} \tag{85}$$

Accordingly, in view of (83), we have

$$\| Dh \|^{n-1} \geqslant \alpha_n \left[|h_r|^{n-1} + (n-1) |h_\omega|^{n-1} \right] \tag{86}$$

Next, we integrate and estimate from below each term on the right hand side by using polar coordinates.

$$\fint_{\mathbb{B}} |h_r|^{n-1} \geqslant \fint_{\mathbb{B}} \big| \, |h|_r \, \big|^{n-1} = n \fint_{\mathbb{S}^{n-1}} \left(\int_0^1 \big| \, r |h|_r \, \big|^{n-1} \, \mathrm{d}r \right) \mathrm{d}\omega \tag{87}$$

$$\fint_{\mathbb{B}} |h_\omega|^{n-1} = n \int_0^1 \left(\fint_{\mathbb{S}^{n-1}} |h_\omega|^{n-1} \, \mathrm{d}\omega \right) r^{n-1} \, \mathrm{d}r \tag{88}$$

Let us temporarily fix r and look at the homeomorphism $\Phi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ given by

$$\Phi(\omega) = \frac{h(r\omega)}{|h(r\omega)|} \qquad \text{for} \quad \omega \in \mathbb{S}^{n-1} \tag{89}$$

The mean Hilbert-Schmidt norm of its tangent map $D\Phi \; : \; \mathbf{T}\mathbb{S}^{n-1} \to \mathbf{T}\mathbb{S}^{n-1}$ can easily be computed as $|D\Phi(r\omega)| = |h_\omega| \, \frac{r}{|h|}$. We emphasize that $|h(x)| = |h(r\omega)|$ depends only on r, not on $\omega \in \mathbb{S}^{n-1}$. We have a point-wise inequality for the pullback of the $(n-1)$-area form on $\mathbb{S}^{n-1}$, $\Phi^\sharp(\Xi) = J(x, \Phi)\, \mathrm{d}\omega \leqslant |D\Phi|^{n-1}\, \mathrm{d}\omega$. Let $R \overset{\text{def}}{=\!=} |h(r\omega)|$. Now, the degree formula yields

$$\begin{aligned} \fint_{\mathbb{S}^{n-1}} |h_\omega|^{n-1}\, \mathrm{d}\omega \;&=\; \frac{R^{n-1}}{r^{n-1}} \fint_{\mathbb{S}^{n-1}} |D\Phi|^{n-1}\; \mathrm{d}\omega \geqslant \frac{R^{n-1}}{r^{n-1}} \deg \Phi \\ &=\; \frac{R^{n-1}}{r^{n-1}} = \frac{1}{r^{n-1}} \fint_{\mathbb{S}^{n-1}} |h(r\omega)|^{n-1}\, \mathrm{d}\omega \end{aligned} \tag{90}$$

The above computation may be summarized as

$$\fint_{\mathbb{B}} \|\, Dh \,\|^{\,n-1} \geqslant n\, \alpha_n \fint_{\mathbb{S}^{n-1}} \left[\int_0^1 \Big[\big|\, r|h|_r \,\big|^{n-1} + (n-1)|h|^{n-1} \Big] \; \mathrm{d}r \right] \mathrm{d}\omega$$

With the aid of the arithmetic-geometric inequality, we estimate the inside integral as follows,

$$\int_0^1 \Big[\big|\, r|h|_r \,\big|^{n-1} + (n-2)|h|^{n-1} + |h|^{n-1} \Big] \; \mathrm{d}r \geqslant$$

$$\int_0^1 \big[(n-1) r |h|_r |h|^{n-2} + |h|^{n-1} \big] \; \mathrm{d}r = \int_0^1 \big(r|h|^{n-1} \big)' \; \mathrm{d}r = 1$$

for every $\omega \in \mathbb{S}^{n-1}$. Finally, we arrive at the desired estimate in which the right hand side is independent of the mapping h,

$$\fint_{\mathbb{B}} \|\, Dh \,\|^{\,n-1} \geqslant n\, \alpha_n = \begin{cases} \sqrt{2}\,, & \text{if } n = 2 \\ n\,, & \text{if } n \geqslant 3 \end{cases} \tag{91}$$

Incidentally, or not, these estimates are sharp for $n = 2$ and $n = 3$. Indeed. the identity map is a minimizer in these cases. It is not clear what the minimum of the p-energy is if $n - 1 \leqslant p < n$, where $n = 4, 5, \ldots$.

10.1.3 The most general case, proof of Theorem 1.5.

We have seen that if $p \geqslant n - 1$, then the infimum energy among spherical mappings is positive. This is false for general mappings when $p = n - 1 \geqslant 2$.

The proof of Theorem 1.5 is surprisingly involved, so we split it into two parts. The first rather easy case, deals with the positive infimum energy.

Part 1. Assume that $p > n-1$ for $n = 3, 4, \ldots$ and $p \geqslant 1$ for $n = 2$. The statement of Theorem 1.5 is immediate in conjunction with an estimate for monotone functions.

Definition 10.1. Let Ω be a domain in $\mathbb{R}^n$. A continuous function $u : \Omega \to \mathbb{R}$ is said to be monotone [51] if for every compact $\mathbb{F} \subset \Omega$, we have

$$\min_{\partial \mathbb{F}} u = \min_{\mathbb{F}} u \leqslant \max_{\mathbb{F}} u = \max_{\partial \mathbb{F}} u \tag{92}$$

In other words, u satisfies both maximum and minimum principle.[2]

Proposition 10.1. (Reverse isoperimetric bound) *Suppose $u \in \mathscr{W}^{1,p}(\mathbb{B}) \cap \mathscr{C}(\mathbb{B})$ is monotone and vanishes at the origin.* [3] *Then*

$$\fint_{\partial \mathbb{B}_r} |u| \leqslant C_p(n)\, r \left(\fint_{\mathbb{B}_r} |\nabla u|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \tag{93}$$

for each ball $\mathbb{B}_r = \{x \in \mathbb{R}^n \,;\, |x| < r\}$ with $0 < r < 1$.

That this yields (15) may be seen by applying (93) to every coordinate function of the homeomorphism h, indeed:

$$\left(\fint_{\mathbb{B}} \| Dh(x) \|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \geqslant C \sum_{i=1}^{n} \fint_{\partial \mathbb{B}} |h^i| \geqslant C \fint_{\partial \mathbb{B}} |h| = C \fint_{\partial \mathbb{B}} 1 = C$$

as desired. The reader may wish to observe that the averages of the coordinate functions need not be large, but their sum does.

Proof of (93). It is a simple matter of integration by parts to obtain

$$\begin{aligned} \int_{\mathbb{B}_r} |\nabla u| \;\; & \geqslant \;\; \int_{\mathbb{B}_r} \big|\nabla |u|\big| \geqslant \int_{\mathbb{B}_r} \frac{\partial |u|}{\partial \rho} = \int_{\mathbb{S}^{n-1}} \left(\int_0^r \rho^{n-1} \frac{\partial |u|}{\partial \rho} \, \mathrm{d}\rho \right) \mathrm{d}\omega \\ & = \;\; \int_{\mathbb{S}^{n-1}} \left[r^{n-1} \big|u(r\omega)\big| - (n-1) \int_0^r \rho^{n-2} \big|u(\rho\omega)\,\big| \mathrm{d}\rho \right] \mathrm{d}\omega \\ & = \;\; \int_{\partial \mathbb{B}_r} |u| - (n-1) \int_{\mathbb{B}_r} \frac{|u(x)|}{|x|} \, \mathrm{d}x \end{aligned}$$

Hence,

$$\fint_{\partial \mathbb{B}_r} |u| \leqslant \frac{r}{n} \fint_{\mathbb{B}_r} |\nabla u| + \frac{n-1}{n}\, r \fint_{\mathbb{B}_r} \frac{|u(x)|}{|x|} \, \mathrm{d}x \tag{94}$$

---

[2] For Sobolev functions the more relevant concept is that of weak monotonicity [54]. But we do not enter into such generalities as the need will not arise, see [27, 37, 73, 53].

[3] The monotonicity assumption can be relaxed. The only property needed is that u assumes value zero on each sphere $\partial \mathbb{B}_\rho$, for $0 < \rho < 1$. Monotonicity certainly yields this property, and the coordinate functions of a homeomorphism are monotone.

This latter inequality is in fact sharp, equality holds when $u(x) = |x|$. We aim to estimate each term in the right hand side by means of the $\mathscr{L}^p$-norm of the gradient. The first integral is easily handled by Hölder's inequality

$$\fint_{\mathbb{B}_r} |\nabla u| \leqslant \left(\fint_{\mathbb{B}_r} |\nabla u|^p \right)^{\frac{1}{p}} \tag{95}$$

For the second integral we appeal to the well known oscillation inequality on spheres, [4]

$$\operatorname*{osc}_{\partial \mathbb{B}_\rho} u \;\leqslant\; A_p(n)\, \rho \left(\fint_{\partial \mathbb{B}_\rho} |\nabla u|^p \right)^{\frac{1}{p}} \qquad 0 < \rho < r \tag{96}$$

where $p > n-1$ if $n = 3, 4, ...$ and $p \geqslant 1$ if $n = 2$. It is crucial at this point that u, being monotone, assumes value zero on each sphere $\partial \mathbb{B}_\rho$. Thus

$$|u(x)| \;\leqslant\; \operatorname*{osc}_{\partial \mathbb{B}_\rho} u \;\leqslant\; A_p(n)\, \rho \left(\fint_{\partial \mathbb{B}_\rho} |\nabla u|^p \right)^{\frac{1}{p}} \quad \text{for all } \; x \in \partial \mathbb{B}_\rho$$

Therefore, we obtain

$$\begin{aligned} \int_{\mathbb{B}_r} \frac{|u(x)|}{|x|}\, \mathrm{d}x \;&\leqslant\; \int_0^r \left(\int_{\partial \mathbb{B}_\rho} |u| \right) \frac{\mathrm{d}\rho}{\rho} \\ &\leqslant\; A_p(n)\, (\omega_{n-1})^{\frac{p-1}{p}} \int_0^r \rho^{\frac{(n-1)(p-1)}{p}} \left(\int_{\partial \mathbb{B}_\rho} |\nabla u|^p \right)^{\frac{1}{p}} \mathrm{d}\rho \qquad (97) \\ &\leqslant\; A_p(n)\, (\omega_{n-1})^{\frac{p-1}{p}} \left(\int_0^r \rho^{n-1}\, \mathrm{d}\rho \right)^{\frac{p-1}{p}} \left(\int_0^r \left(\int_{\partial \mathbb{B}_\rho} |\nabla u|^p \right) \mathrm{d}\rho \right)^{\frac{1}{p}} \end{aligned}$$

Hence,

$$\fint_{\mathbb{B}_r} \frac{|u(x)|}{|x|}\, \mathrm{d}x \;\leqslant\; A_p(n) \left(\fint_{\mathbb{B}_r} |\nabla u|^p \right)^{\frac{1}{p}}$$

Finally, substituting (95) and (97) into (94), we arrive at the inequality

$$\fint_{\partial \mathbb{B}_r} |u| \leqslant C_p(n)\, r \left(\fint_{\mathbb{B}_r} |\nabla u|^p \, \mathrm{d}x \right)^{\frac{1}{p}}$$

which is what we wished to establish.

Part 2. Assume that $1 \leqslant p \leqslant n-1$ and $n \geqslant 3$.

We shall construct homeomorphisms $h_k : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}$ of the unit ball $\mathbb{B} \subset \mathbb{R}^n$, $h_k(0) = 0$, in the Sobolev space $\mathscr{W}^{1,n-1}(\mathbb{B}\,, \mathbb{B})$ having arbitrarily small p-harmonic energy. By Hölder's inequality, it suffices to show that

$$\lim_{k \to \infty} \int_{\mathbb{B}} \| \, Dh_k(x) \, \|^{\,n-1}\, \mathrm{d}x = 0\,, \qquad n \geqslant 3 \tag{98}$$

---

[4] For the best constant $A_p(n)$, see (27).

As we have learnt from Theorem 10.2 such sequence cannot be found in the class of spherical mappings. A close look at the arguments in Section 10.1.2 reveals that h_k must take spheres $\mathbb{S}_r = \{x \in \mathbb{R}^n\, ;\ |x| = r\}$ into surfaces of very small $(n-1)$-dimensional area. They are additionally required to take most of the sphere $\mathbb{S}_r$ into a small neighborhood of one point. Thus, we consider deformations of the form

$$h(x) = \big(\, B(r)\cos\Phi(\theta)\, ,\ A(r)\mathfrak{s}\sin\Phi(\theta)\,\big) \tag{99}$$

where we use spherical coordinates, see Section 2.1

$$x = \big(\, r\cos\theta\, ,\ r\mathfrak{s}\sin\theta\,\big)\, ,\ \ \mathfrak{s}\in\mathbb{S}^{n-2}\, ,\ 0 \leqslant r \leqslant 1\, ,\ 0 \leqslant \theta \leqslant \pi \tag{100}$$

To ensure that h is a homeomorphism we shall contract functions A, B which continuously increase from 0 to 1, for $0 \leqslant r \leqslant 1$, while the meridian angle $\Phi = \Phi(\theta)$ will continuously increase from 0 to π as $0 \leqslant \theta \leqslant \pi$. With these properties each sphere $\mathbb{S}_r$ is mapped onto an ellipsoid

$$\frac{x_1^2}{A^2} + \cdots \frac{x_{n-1}^2}{A^2} + \frac{x_n^2}{B^2} = 1$$

Clearly h maps the unit ball $\mathbb{B}$ onto itself and keeps the origin fixed. The Hilbert-Schmidt norm of the differential is computed as

$$\begin{aligned} \|\, Dh(x)\,\|^{\,2} &= \dot{B}^2\cos^2\Phi + \dot{A}^2\sin^2\Phi + \\ &+ \frac{1}{r^2}\left(B^2\sin^2\Phi + A^2\cos^2\Phi\right)\dot{\Phi}^2 \\ &+ (n-2)\frac{A^2\sin^2\Phi}{r^2\sin^2\theta} \end{aligned} \tag{101}$$

As before, we conveniently enlarge the class of contenders. We say that a continuous mapping $h : \mathbb{B} \to \mathbb{B}$ is admissible if it is a limit in $\mathscr{W}^{1,n-1}(\mathbb{B}\, , \mathbb{B})$ of deformations $\mathbb{B}$ onto itself, $h(0) = 0$. Then (98) is equivalent to

$$\inf \int_{\mathbb{B}} \|\, Dh(x)\,\|^{\,n-1}\, \mathrm{d}x = 0\, , \qquad n \geqslant 3 \tag{102}$$

where the infimum runs over all admissible mappings. Fix a positive integer k for a moment. We define an admissible mapping $h_k : \mathbb{B} \to \mathbb{B}$ by setting

$$\begin{aligned} A_k(r) &= \begin{cases} 0\, , & \text{for } 0 \leqslant r \leqslant 1 - k^{\frac{n}{1-n}} \\ k^{\frac{n}{n-1}} r + 1 - k^{\frac{n}{n-1}}\, , & \text{for } 1 - k^{\frac{n}{1-n}} \leqslant r \leqslant 1 \end{cases} \\ B_k(r) &= \begin{cases} kr\, , & \text{for } 0 \leqslant r \leqslant \frac{1}{k} \\ 1\, , & \text{for } \frac{1}{k} \leqslant r \leqslant 1 \end{cases} \\ \Phi_k(\theta) &= \begin{cases} 0\, , & \text{for } 0 \leqslant \theta \leqslant \frac{1}{k}e^{-k} \\ \pi + \frac{\pi}{k}\log k\theta\, , & \text{for } \frac{1}{k}e^{-k} \leqslant \theta \leqslant \frac{1}{k} \\ \pi & \text{for } \frac{1}{k} \leqslant \theta \leqslant \pi \end{cases} \end{aligned}$$

These functions are not strictly increasing. Nevertheless, each h_k so defined is admissible. In fact h_k is a Lipschitz map. This can be verified by suitable approximation of A_k, B_k and Φ_k. Now comes a computation. We begin with the point-wise inequality.

$$\| Dh_k(x) \|^{n-1} \preccurlyeq \left|\dot{B}_k\right|^{n-1} + \left|\dot{A}_k \sin\Phi_k\right|^{n-1} + \left|\frac{1}{r}\dot{\Phi}_k\right|^{n-1} + \left|\frac{A_k \sin\Phi_k}{r \sin\theta}\right|^{n-1}$$

where the implied constant is independent of k. We estimate $\int_{\mathbb{B}} \| Dh_k(x) \|^{n-1}\, \mathrm{d}x$ in four steps.

First integral:

$$\begin{aligned} \int_{\mathbb{B}} \left|\dot{B}_k\right|^{n-1} &= \omega_{n-1}\int_0^1 \left|\dot{B}_k(r)\right|^{n-1} r^{n-1}\,\mathrm{d}r \\ &= \omega_{n-1}k^{n-1}\int_0^{\frac{1}{k}} r^{n-1}\,\mathrm{d}r = \frac{\omega_{n-1}}{k\,n} \to 0 \end{aligned}$$

as k increases to infinity.

Second integral:

$$\begin{aligned} \int_{\mathbb{B}} \left|\dot{A}_k \sin\Phi_k\right|^{n-1} &= \omega_{n-2}\int_0^1 \left|\dot{A}_k(r)\right|^{n-1} r^{n-1}\,\mathrm{d}r \int_0^{\pi} \sin^{n-1}\Phi_k\, \sin^{n-2}\theta\,\mathrm{d}\theta \\ &\leqslant k^n\left(\int_{1-k^{\frac{1}{1-n}}}^1 r^{n-1}\,\mathrm{d}r\right)\left(\omega_{n-2}\int_0^{\frac{1}{k}} \sin^{n-2}\theta\,\mathrm{d}\theta\right) \\ &\leqslant \frac{\omega_{n-2}}{\sqrt[n-1]{k}} \to 0 \end{aligned}$$

as k increases to infinity.

Third integral:

$$\begin{aligned} \int_{\mathbb{B}} \left|\frac{1}{r}\dot{\Phi}_k\right|^{n-1} &= \omega_{n-2}\int_0^{\pi} \left|\dot{\Phi}_k\right|^{n-1} \sin^{n-2}\theta\,\mathrm{d}\theta \\ &= \omega_{n-2}\int_{\frac{1}{k}e^{-k}}^{\frac{1}{k}} \frac{\pi^{n-1}}{k^{n-1}\,\theta^{n-1}} \sin^{n-2}\theta\,\mathrm{d}\theta \\ &\leqslant \frac{\pi^{n-1}\omega_{n-2}}{k^{n-1}}\int_{\frac{1}{k}e^{-k}}^{\frac{1}{k}} \frac{\mathrm{d}\theta}{\theta} = \frac{\pi^{n-1}\omega_{n-2}}{k^{n-2}} \to 0 \end{aligned}$$

as k increases to infinity.

Forth integral:

$$\begin{aligned}
\int_{\mathbb{B}} \left| \frac{A_k\, \sin \Phi_k}{r\, \sin\theta} \right|^{n-1} &= \omega_{n-2} \left(\int_0^1 A_k^{n-1}(r)\, \mathrm{d}r \right) \left(\int_0^\pi \frac{\sin^{n-1}\Phi_k}{\sin\theta}\, \mathrm{d}\theta \right) \\
&\leqslant \frac{\pi\, \omega_{n-2}}{2\, k^{\frac{n}{n-1}}} \int_{\frac{1}{k}e^{-k}}^{\frac{1}{k}} \sin^{n-1}\left(\frac{\pi}{k} \log \frac{1}{k\theta} \right) \frac{\mathrm{d}\theta}{\theta} \\
&= k\, \frac{\omega_{n-2}}{2\, k^{\frac{n}{n-1}}} \int_0^\pi \sin^{n-1} t\, \mathrm{d}t \\
&= \frac{\omega_{n-1}}{2\, \sqrt[n-1]{k}} \to 0
\end{aligned}$$

as k increases to infinity.

The computation is complete, showing that

$$\lim_{k\to\infty} \int_{\mathbb{B}} \| \, Dh_k(x)\, \|^{\, n-1}\, \mathrm{d}x = 0\,, \qquad n \geqslant 3$$

as desired.

References

[1] Aleksandrov, P. S., and Pasynkov, B. A. *Vvedenie v teoriyu razmernosti: Vvedenie v teoriyu topologicheskikh prostranstv i obshchuyu teoriyu razmernosti.* (Russian) [Introduction to dimension theory: An introduction to the theory of topological spaces and the general theory of dimension] Izdat. "Nauka", Moscow, 1973.

[2] Alexander, J.W. *An Example of a Simply Connected Surface Bounding a Region which is not Simply Connected.* Proceedings of the National Academy of Sciences 1924; 10(1): 8–10.

[3] Antman, S. S. *Fundamental mathematical problems in the theory of nonlinear elasticity.* Univ. Maryland, College Park, Md., (1975), 35–54.

[4] Astala, K., Iwaniec, T., Martin, G. J., and Onninen, J. *Extremal mappings of finite distortion.* Proc. London Math. Soc. (3) **91** (2005), no. 3, 655–702.

[5] Ball, J.M. *Convexity conditions and existence theorems in nonlinear elasticity.* Arch. Rational Mech. Anal. **63** (1978), 337–403.

[6] Ball, J.M. *Global invertibility of Sobolev functions and the interpenetration of matter.* Proc. Roy. Soc. Edinburgh Sect. A **88**, no. 3-4, (1981), 315–328.

[7] Ball, J.M. *Discontinuous equilibrium solutions and cavitation in nonlinear elasticity.* Philos. Trans. Roy. Soc. London Ser. A **306** (1982), no. 1496, 557–611.

[8] Ball, J.M. Private communication, Ischia (2007).

[9] Bethuel, F. (1991). *The approximation problem for Sobolev maps between two manifolds.* Acta Math. **167** no. 3-4, 153–206.

[10] Bojarski, B., and Iwaniec, T. *Analytical foundations of the theory of quasiconformal mappings in R^n.* Ann. Acad. Sci. Fenn. Ser. A I Math. **8** (1983), no. 2, 257–324.

[11] Borsuk, K. Über Schnitte der n-dimensionalen Euklidischen Räume. (German) Math. Ann. **106** (1932), no. 1, 239–248.

[12] Borsuk, K. Über Schnitte der n-dimensionalen Euklidischen Räume. Math. Ann. 106 (1932), no. 1, 239–248.

[13] Borsuk, K. *Drei Sätze über die n-dimensionale Euklidische Sphäre.* Fund. Math., **20** (1933), 177–190.

[14] Borsuk, K. *Set theoretical approach to the disconnection theory of the Euclidean space.* Fund. Math. **37**, (1950). 217–241.

[15] Brezis, H., and Nirenberg, L. *Degree theory and BMO, Part I: Compact manifolds without boundaries.* Selecta Math. **1** (1995) 197-263.

[16] Brezis, H., and Nirenberg, L. *Degree theory and BMO. II. Compact manifolds with boundaries.* Selecta Math. (N.S.) **2** , no. 3, (1996), 309-368.

[17] Brouwer, L. E. J. *Beweis des Jordanschen Satzes fr den n-dimensionalen Raum.* Math. Ann. 71 (1911), no. 3, 314–319.

[18] Brown, M. *A proof of the generalized Schoenflies theorem.* Bull. Amer. Math. Soc., vol. 66, (1960) 74–76.

[19] Ciarlet, P. G. *Mathematical elasticity.* Vol. I. Three-dimensional elasticity. Studies in Mathematics and its Applications, 20. North-Holland Publishing Co., Amsterdam 1988.

[20] Ciarlet, P. G., and Nečas, J. *Injectivity and self-contact in nonlinear elasticity.* Arch. Rational Mech. Anal. **97** (1987), no. 3, 171–188.

[21] Csörnyei, M., Hencl, S, and Malý, J. *Homeomorphisms in the Sobolev space* $W^{1,n-1}$. Preprint MATH-KMA-2007/252, Charles University in Prague.

[22] Duren, P. *Harmonic mappings between planar domains*, Cambridge Tracts in Mathematics, 156. Cambridge University Press, Cambridge, 2004.

[23] Engelking, R., and Sieklucki, K. *Topology: a geometric approach.* Sigma Series in Pure Mathematics, 4. Heldermann Verlag, Berlin, 1992.

[24] Federer, H. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band **153** Springer-Verlag New York Inc., New York (1969).

[25] Gehring, F.W. *Rings and quasiconformal mappings in space.* Trans. Amer. Math. Soc. **103**, (1962) 353-393.

[26] Gehring, F. W., and Väisälä, J. *The coefficients of quasiconformality of domains in space.* Acta Math. **114**, (1965), 1–70.

[27] Gol'dstein V.M., and Vodop'yanov S.K. *Quasiconformal mappings and spaces of functions with generalized first derivatives.* Sibirsk. Mat. Z. **17** (1976), 515-531.

[28] Hajłasz P., Iwaniec T., Malý J., and Onninen J. *Weakly differentiable mappings between manifolds.* Mem. Amer. Math. Soc., to appear.

[29] Hajłasz, P., and Malý, J. *Approximation in Sobolev spaces of nonlinear expressions involving the gradient.* Ark. Mat. **40** (2002), no. 2, 245–274.

[30] Hang, F., and Lin, F. *Topology of Sobolev mappings II.* Acta Math. **191**, no. 1, (2003), 55–107.

[31] Heinonen, J., and Kilpeläinen, T. *BLD-mappings in* $W^{2,2}$ *are locally invertible.* Math. Ann. **318** (2000), no. 2, 391–396.

[32] Heinonen, J., Kilpeläinen, T., and Martio, O. *Nonlinear potential theory of degenerate elliptic equations.* Oxford Mathematical Monographs, (1993).

[33] Hencl, S., and Koskela, P. *Regularity of the inverse of a planar Sobolev homeomorphism.* Arch. Rational Mech. Anal., to appear.

[34] Hencl, S. Koskela, P., and Onninen, J. *A note on extremal mappings of finite distortion.* Math. Res. Lett. **12** (2005), no. 2-3, 231–237.

[35] Hurewicz, W., and Wallman, H. *Dimension Theory.* Princeton Mathematical Series, v. 4. Princeton University Press, Princeton, N. J., 1941.

[36] Iwaniec, T. *p-harmonic tensors and quasiregular mappings.* Ann. of Math. (2) **136** (1992), no. 3, 589–624.

[37] Iwaniec, T., Koskela, P., and Onninen, J. *Mappings of finite distortion: monotonicity and continuity.* Invent. Math. 144 (2001), no. **3**, 507–531.

[38] Iwaniec, T., and Martin, G. *Quasiregular mappings in even dimensions.* Acta Math. **170** (1993), no. 1, 29–81.

[39] Iwaniec, T., and Martin, G. J. *Geometric Function Theory and Non-linear Analysis.* Oxford Mathematical Monographs, 2001.

[40] Iwaniec, T., and Onninen, J. *n-Harmonic mappings between annuli.* Submitted for publication.

[41] Iwaniec, T., and Onninen, J. *Hyperelastic deformations of smallest total energy.* Submitted for publication.

[42] Joshi, K.D. *A non-symmetric generalization of the Borsuk-Ulam theorem.* Fund. Math. **80** (1973), no. 1, 13–33.

[43] Kauhanen, J., Koskela, P., and Malý, J. *Mappings of finite distortion: condition N.* Michigan Math. J. 49 (2001), no. **1**, 169–181.

[44] Kauhanen, J., Koskela, P., and Malý, J. *Mappings of finite distortion: discreteness and openness.* Arch. Ration. Mech. Anal. 160 (2001), no. 2, 135–151.

[45] Koskela, P., and Malý, J. *Mappings of finite distortion: the zero set of the Jacobian.* J. Eur. Math. Soc. (JEMS) 5 (2003), no. 2, 95–105.

[46] Koskela, P., and Onninen, J., *Mappings of finite distortion: Capacity and modulus inequalities.* J. Reine Angew. Math. 599 (2006), 1–26.

[47] Kuratowski, K. *Remark on an invariance theorem.* Fund. Math. **37**, (1950). 251–252.

[48] Kuratowski, K. *Topology. Vol. I.* Academic Press, New York-London; Państwowe Wydawnictwo Naukowe, olish Scientific Publishers, Warsaw. (1966).

[49] Kuratowski, K. *Topology. Vol. II.* Academic Press, New York-London; Państwowe Wydawnictwo Naukowe, Polish Scientific Publishers, Warsaw (1968).

[50] Lafont, J. F. *Strong Jordan separation and applications to rigidity.* J. London Math. Soc. (2) 73 (2006), no. 3, 681–700.

[51] Lebesgue, H. *Sur le problème de Dirichlet.* Rend. Circ. Palermo **27** (1907), 371-402.

[52] Liouville, J. *Théorème sur l' équation* $\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2 = \lambda(\mathrm{d}\alpha^2+\mathrm{d}\beta^2+\mathrm{d}\gamma^2)$. J. Math Pures Appl., **1**, 15 (1850).

[53] Malý, J., and Martio, O. *Lusin's condition (N) and mappings of the class* $W^{1,n}$. J. Reine Angew. Math. **458** (1995), 19–36.

[54] Manfredi, J. J. *Weakly monotone functions.* J. Geom. Anal. **4** (1994), no. 3, 393–402.

[55] Martio, O., Rickman, S., and Väisälä, J. *Definitions for quasiregular mappings.* Ann. Acad. Sci. Fenn. Ser. A I No. **448** (1969).

[56] Martio, O., Rickman, S., and Väisälä, J. *Distortion and singularities of quasiregular mappings.* Ann. Acad. Sci. Fenn. Ser. A I No. 465 (1970).

[57] Martio, O., Rickman, S., and Väisälä, J. *Topological and metric properties of quasiregular mappings.* Ann. Acad. Sci. Fenn. Ser. A I No. 488 (1971)

[58] Mazur, B. *On embeddings of spheres.* Bull. Amer. Math. Soc. 65 1959 59–65.

[59] McAuley, L. F. *The Proceedings of the First Conference on Monotone Mappings and Open Mappings* Edited by Louis F. McAuley. State University of New York at Binghamton, 1971.

[60] Morrey, C.B. *The Topology of (Path) Surfaces.* Amer. J. Math. **57** (1935), no. 1, 17–50.

[61] Morse, M. *A reduction of the Schoenflies extension problem.* Bull. Amer. Math. Soc. 66 1960 113–115.

[62] Munkholm, H. J. *A Borsuk-Ulam theorem for maps from a sphere to a compact topological manifold.* Illinois J. Math. **13** (1969) 116–124.

[63] Müller, S., and Spector, S. J. *An existence theory for nonlinear elasticity that allows for cavitation.* Arch. Rational Mech. Anal. **131** (1995), no. 1, 1–66.

[64] Müller, S., Spector, S. J., and Tang, Q. *Invertibility and a topological property of Sobolev maps.* SIAM J. Math. Anal. **27** (1996), no. 4, 959–976.

[65] Onninen, J. *Regularity of the inverse of spatial mappings with finite distortion*, Calc. Var. Partial Differential Equations 26 (2006), no. 3, 331–341.

[66] Reshetnyak, Yu. G. *Bounds on moduli of continuity for certain mappings.* (Russian). Sibirsk. Mat. Sh. **7** (1966), 1106-1114.

[67] Reshetnyak, Yu. G. *Space mappings with bounded distortion.* Translations of Mathematical Monographs, 73. American Mathematical Society, Providence, RI, 1989.

[68] Rickman, S. *Quasiregular mappings.* Springer-Verlag, Berlin, 1993.

[69] Schoen, R, and Uhlenbeck, K. *A regularity theory for harmonic maps.* J. Differential Geom. 17, no. 2, (1982), 307-335.

[70] Schoen, R., and Uhlenbeck, K. *Boundary regularity and the Dirichlet problem for harmonic maps.* J. Differential Geom. 18, no. 2, (1983), 253-268.

[71] Schottky, F.H. *Über konforme Abbildung von mehrfach zusammenhängenden Fläche.* J. für Math., **83**, (1877).

[72] Sivaloganathan, J., and Spector, S. J. *Myriad radial cavitating equilibria in nonlinear elasticity.* SIAM J. Appl. Math. **63** (2003), no. 4, 1461–1473.

[73] Šverák, V. *Regularity properties of deformations with finite energy.* Arch. Rational Mech. Anal. **100** (1988), no. 2, 105–127.

[74] Tang, Q. *Almost-everywhere injectivity in nonlinear elasticity.* Proc. Roy. Soc. Edinburgh Sect. A **109** (1988), no. 1-2, 79–95.

[75] Yang, C-T. *On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson. I.* Ann. of Math. (2) **60**, (1954). 262–282.

[76] Vuorinen, M. *Conformal geometry and quasiregular mappings.* Lecture Notes in Mathematics, 1319. Springer-Verlag, Berlin, 1988.

[77] Väisälä, J. *Lectures on n-dimensional quasiconformal mappings.* Lecture Notes in Mathematics, Vol. **229**. Springer-Verlag, Berlin-New York, (1971).

Department of Mathematics, Syracuse University, Syracuse, NY 13244,
USA
E-mail address: **tiwaniec@mailbox.syr.edu**

Department of Mathematics, Syracuse University, Syracuse, NY 13244,
USA
E-mail address: **jkonnine@syr.edu**